An Alphabetical Approach to the Nivat's Conjecture

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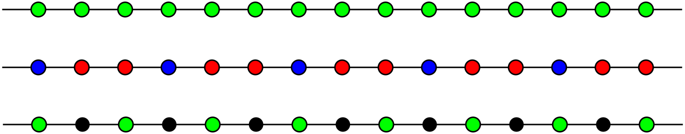
- The unidimensional case
- On the Nivat's Conjecture
- Expansive subdynamics
- Generating sets
- Expansive directions
- A connection of nonexpasive direction with periodicity
- Our main result

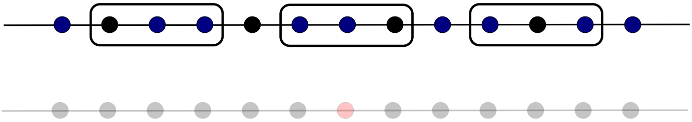
Fixed a finite alphabet $\mathcal{A}$, let $\mathcal{A}^{\mathbb{Z}}$ to be the product space. A sequence $\xi \in \mathcal{A}^{\mathbb{Z}}$ has the form $(\xi_i)_{i \in \mathbb{Z}}$, where $\xi_i \in \mathcal{A}$ for all $i \in \mathbb{Z}$.

Definition

A sequence $\xi = (\xi_i)_{i \in \mathbb{Z}}$ is said to be periodic of period $m \geq 1$ if $\xi_{i+m} = \xi_i$ for all $i \in \mathbb{Z}$.



Given a sequence $\xi \in \mathcal{A}^{\mathbb{Z}}$, the *n*-complexity of ξ, denoted by $P_{\xi}(n)$, is defined to be the number of distinct words of the form $\xi_i \xi_{i+1} \cdots \xi_{i+n-1}$ appearing in ξ.



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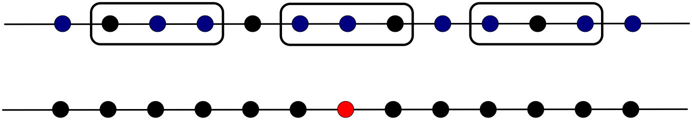

Theorem (Morse-Hedlund [6])

For a sequence $\xi \in \mathcal{A}^{\mathbb{Z}}$, if $P_\xi(n) \leq n$ for some $n \in \mathbb{N}$, then ξ is periodic of period at most n.

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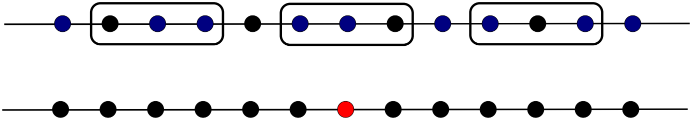

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Since techniques used to address the Nivat's conjecture usually relies on Morse-Hedlund's Theorem, an improved version of this classical result may mean a new step towards a proof for the conjecture.

Theorem (Alphabetical Morse-Hedlund's Theorem)

For a sequence $\xi \in \mathcal{A}^{\mathbb{Z}}$ making use of all colors of $\mathcal{A}$, if there exists $n \in \mathbb{N}$ such that

$$P_\xi(n) \leq n + |\mathcal{A}| - 2,$$

then ξ is periodic of period at most $n + |\mathcal{A}| - 2$.

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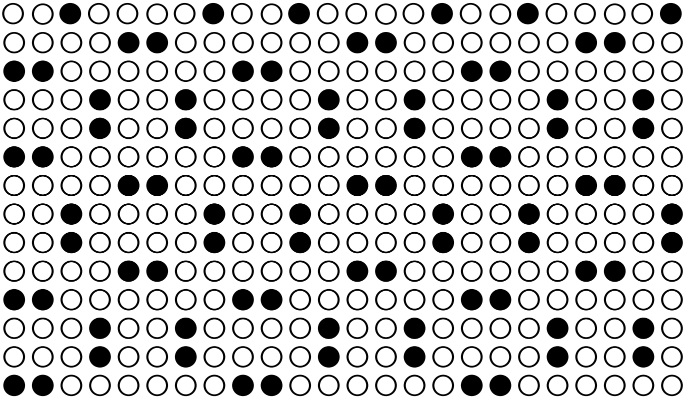
Fixed a finite alphabet $\mathcal{A}$, let $\mathcal{A}^{\mathbb{Z}^d}$ to be the product space. A configuration $\eta \in \mathcal{A}^{\mathbb{Z}^d}$ has the form $(\eta_g)_{g\in\mathbb{Z}^d}$, where $\eta_g \in \mathcal{A}$ for all $g \in \mathbb{Z}^d$.

Definition

A configuration $\eta = (\eta_g)_{g\in\mathbb{Z}^d}$ is said to be periodic of period $h \in (\mathbb{Z}^d)^*$ if $\eta_{g+h} = \eta_g$ for all $g \in \mathbb{Z}^d$.



The complexity function $P_\eta(n,k)$ is defined to be the number of distinct $n \times k$ rectangles of symbols appearing in η.

Conjecture (Nivat's Conjecture [7], 1997)

For $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, if there are $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq nk$, then η is periodic.

- It is a natural generalization of Morse-Hedlund's Theorem for the two-dimensional case.
- It is not an equivalence.
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Example

For $n \geq 3$*, let* $\eta \in \{0,1\}^{\mathbb{Z}^3}$ *to be the configuration defined by*

$$\eta_g := 1 \textit{ if } g = (i,0,0) \textit{ or } g = (0,i,n) \textit{ for } i \in \mathbb{Z},$$

and $\eta_g := 0$ *otherwise. It is easy to see that*

$$P_\eta(n,n,n) = 2n^2 + 1 < n^3.$$

however, η *is not a periodic configuration.*

- Sander and Tijdeman showed that, if there is $n \in \mathbb{N}$ such that $P_\eta(n,2) \leq 2n$, then $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ is periodic.
- Epifanio, Koskas and Mignosi [4] showed that, if there are $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq \frac{1}{144}nk$, then $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ is periodic.
- Quas and Zamboni [8] showed that, if there are $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq \frac{1}{16}nk$, then $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ is periodic.
- Cyr and Kra [2] showed that, if there is $n \in \mathbb{N}$ such that $P_\eta(n,3) \leq 3n$, then $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ is periodic.
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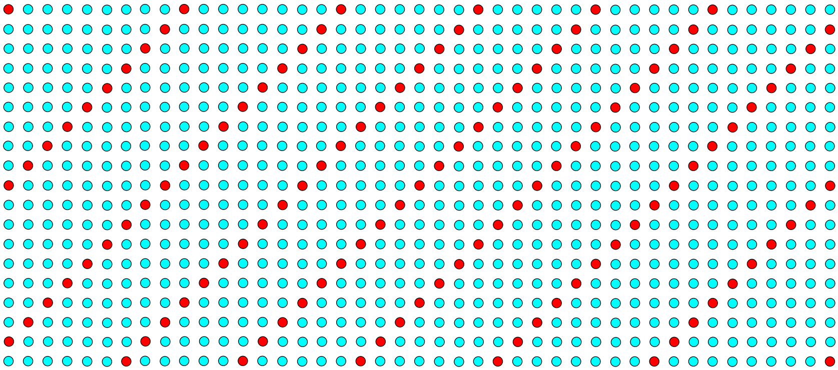
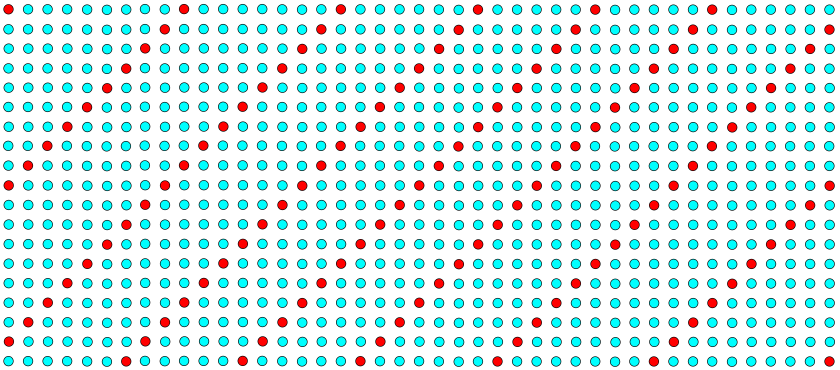
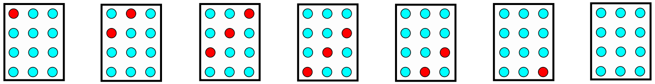
Theorem (Cyr and Kra)

For $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, if there are $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq \frac{1}{2}nk$, then η is periodic.







It is easy to see that

$$P_\eta(n,k) = n + k \qquad \text{if } n + k \leq 7$$

$$P_\eta(n,k) = n + k + \frac{1}{2}(n+k-7)(n+k-6) \qquad \text{otherwise}$$

and therefore

$$P_\eta(n,k) > \frac{1}{2}nk \quad \forall\, n,k \in \mathbb{N}.$$

However, we have

$$P_\eta(3,4) = 7 = \frac{1}{2}(3 \times 4) + 1 = \frac{1}{2}(3 \times 4) + |colors| - 1.$$

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For $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, if there are $n, k \in \mathbb{N}$ such that

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then η is periodic.

Using the notion of expansive subspaces introduced by Boyle and Lind, Cyr and Kra shed a new light towards a proof for Nivat's Conjecture by relating expansive subspaces to periodicity.

Definition

Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift. A subspace F of $\mathbb{R}^d$ is said to be expansive on X if there is $t > 0$ such that

$$\forall\, x, y \in X, \quad x|_{F^t} = y|_{F^t} \quad \Longrightarrow \quad x = y,$$

where $F^t := \{g \in \mathbb{Z}^d : \mathrm{dist}(g, F) \leq t\}$. If a subspace fails to meet this condition, it is called a nonexpansive subspace on X.

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Denote $X_\eta := \overline{Orb(\eta)}$

- Two nonexpansive 1-d subspaces on X_η
  $\Rightarrow \quad \eta \in \mathcal{A}^{\mathbb{Z}^2}$ aperiodic.
- Low complexity and at most one nonexpansive 1-d subspace
  $\Rightarrow \quad \eta \in \mathcal{A}^{\mathbb{Z}^2}$ periodic (Cyr-Kra [3]).
- No nonexpansive 1-d subspaces on X_η
  $\Rightarrow \quad \eta \in \mathcal{A}^{\mathbb{Z}^2}$ 2-periodic (Boyle-Lind [1]).

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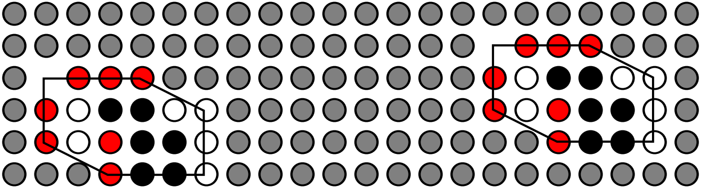
Definition (Cyr and Kra)

Given $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, a point $g \in \mathcal{S}$ is said to be η-generated by $\mathcal{S} \subset \mathbb{Z}^2$ when $P_\eta(\mathcal{S}\backslash\{g\}) = P_\eta(\mathcal{S})$. A finite, nonempty, convex set $\mathcal{S} \subset \mathbb{Z}^2$ for which every vertex is η-generated is called an η-generating set.



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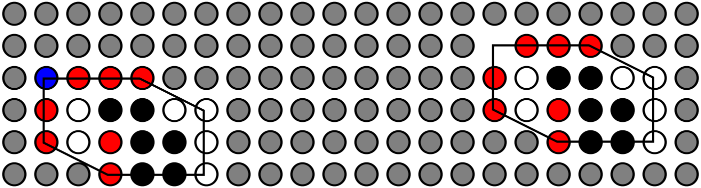

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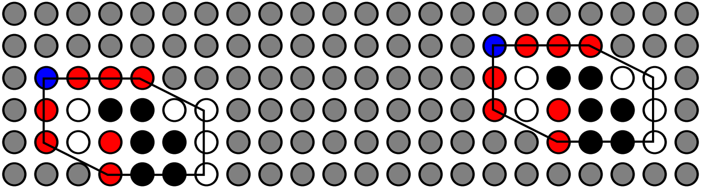

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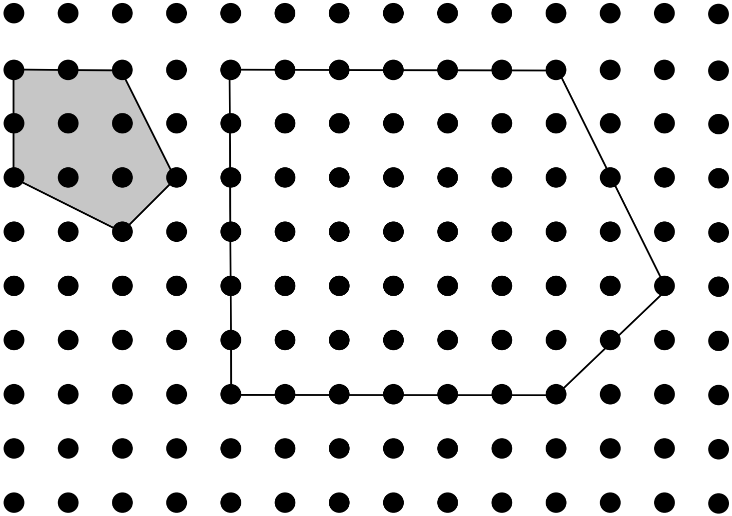

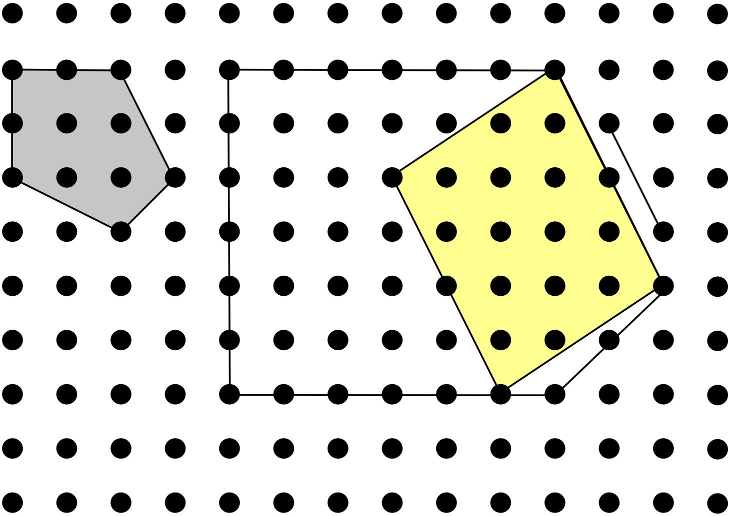

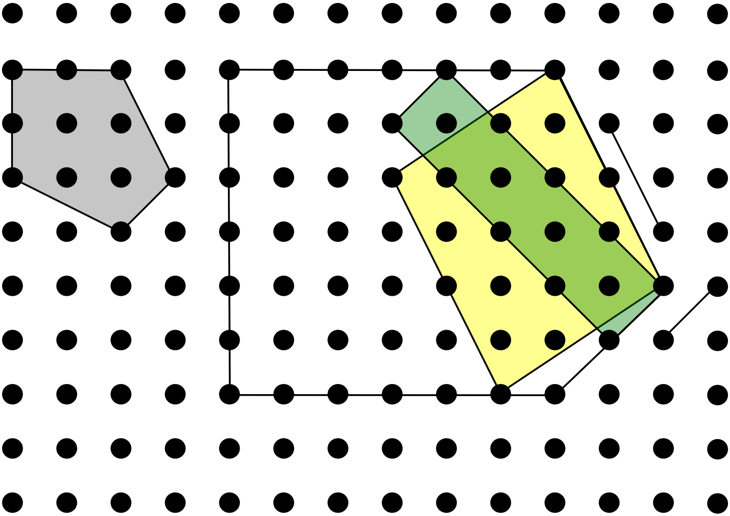

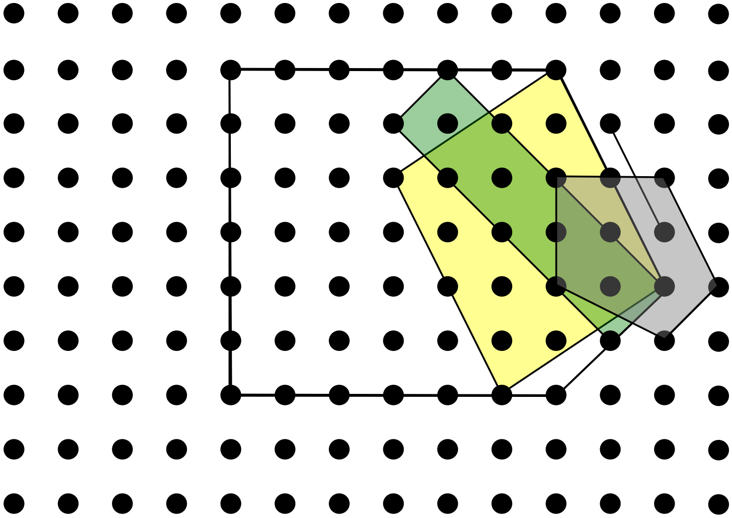

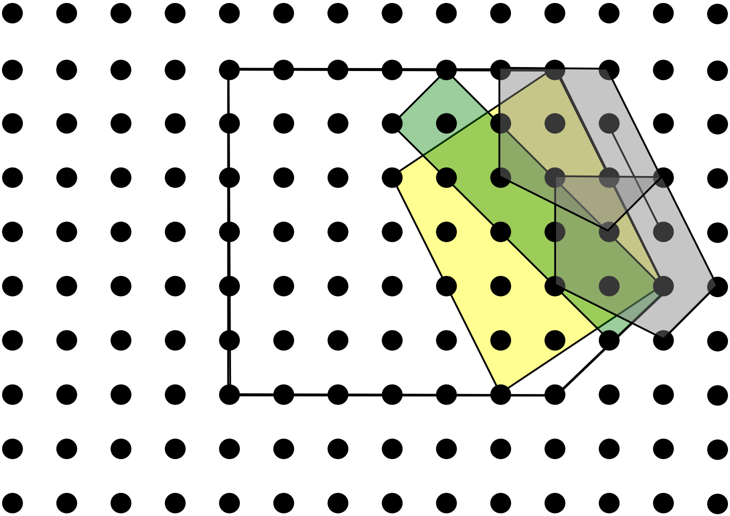

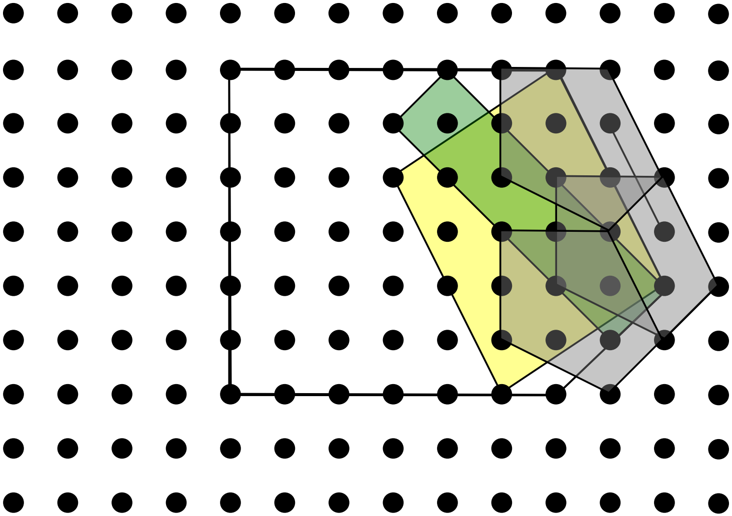

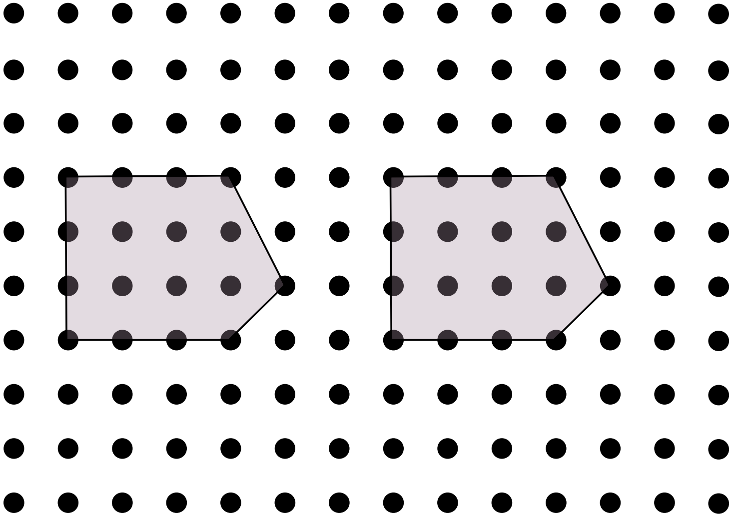

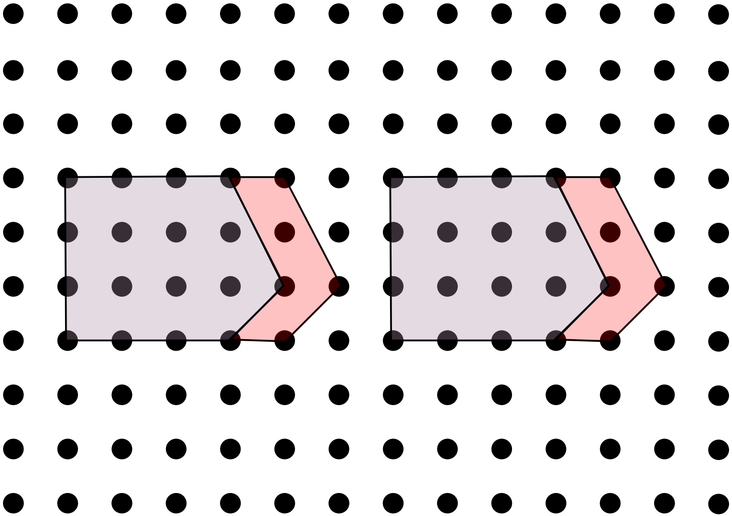

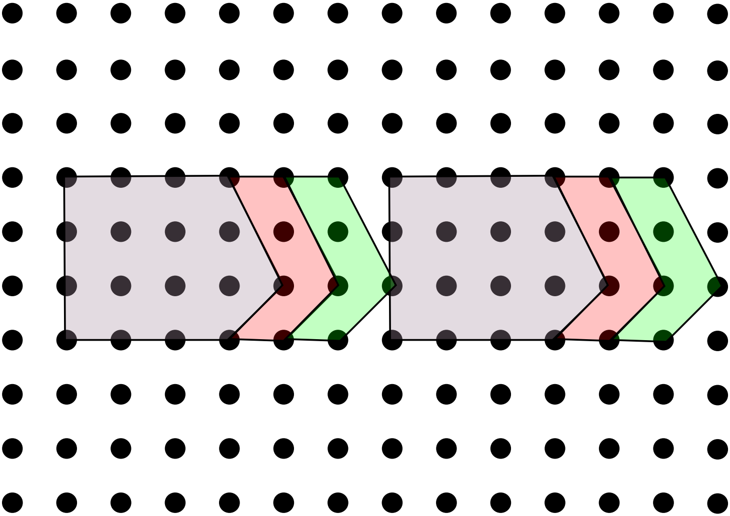

Following Boyle and Lind, we use a suitable notion of expansiveness.

- For an 1-d subspace ℓ of $\mathbb{R}^2$, let $\vec{H}$ and $\overleftarrow{H}$ denote the two closed half-planes of $\mathbb{R}^2$ with boundary ℓ.

Definition

Given $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, let ℓ an 1-d subspace of $\mathbb{R}^2$. We say that $\vec{\ell}$ is an expansive direction on X_η if

$$\forall\, x, y \in X_\eta = \overline{Orb(\eta)}, \quad x|_{\vec{H}\cap\mathbb{Z}^2} = y|_{\vec{H}\cap\mathbb{Z}^2} \implies x = y.$$

If $\vec{\ell}$ fails to meet this condition, it is called a nonexpansive direction on X_η.

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- We can think in $\vec{\ell}$ and $\overleftarrow{\ell}$ as being the oriented edges of the half-planes $\vec{H}$ and $\overleftarrow{H}$ (positively oriented).

Proposition

Suppose $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ is periodic. If $\vec{\ell} \in \mathbb{G}_1$ is a non-expansive direction on X_η, then the oriented line $\overleftarrow{\ell} \in \mathbb{G}_1$ antiparallel to $\vec{\ell}$ is also a non-expansive direction on X_η.

For $x \in X_\eta = \overline{Orb(\eta)}$, a convex set $\mathcal{U} \in \mathcal{F}_\mathcal{C}$ and an oriented line $\vec{\ell} \in \mathbb{G}_1$ through the origin in $\mathbb{R}^2$, let $L_{\vec{\ell}}(\mathcal{U}, x)$ the subfamily of $L(\mathcal{U}, \eta) = \{(T^g \eta)|_\mathcal{U} : g \in \mathbb{Z}^2\}$ defined by

$$L_{\vec{\ell}}(\mathcal{U}, x) := \left\{ (T^{t\vec{v}_\ell} x)|_\mathcal{U} \ : t \in \mathbb{Z} \right\}.$$

Given $\mathcal{U} \in \mathcal{F}_\mathcal{C}^{Vol}$, for an oriented line $\vec{\ell} \in \mathbb{G}_1$ and $\gamma \in L(\mathcal{U} \backslash \vec{\ell}_\mathcal{U}, \eta)$, we put

$$N_{\vec{\ell},\mathcal{U}}(\gamma) = |\{\gamma' \in L(\mathcal{U}, \eta) : \gamma'|_{\mathcal{U} \backslash \vec{\ell}_\mathcal{U}} = \gamma\}|.$$

If $\vec{\ell} \in \mathbb{G}_1$ is a non-expansive direction on X_η and $\mathcal{U} \in \mathcal{F}_{\mathcal{C}}^{Vol}$ is an η-generating set, then there is configuration $x \in X_\eta$ such that

$$N_{\vec{\ell},\mathcal{U}}(\gamma) > 1 \quad \forall\, \gamma \in L_{\vec{\ell}}(\mathcal{U} \backslash \vec{\ell}_{\mathcal{U}}, x). \tag{1}$$

The set formed by the configurations $x \in X_\eta = \overline{Orb(\eta)}$ that satisfying (1) is denoted by $\mathcal{N}(\vec{\ell}, \mathcal{U})$.

For any $x \in \mathcal{N}(\vec{\ell}, \mathcal{U})$, from the equality

$$P_\eta(\mathcal{U}) - P_\eta(\mathcal{U} \backslash \vec{\ell}_\mathcal{U}) = \sum_{\gamma \in L(\mathcal{U} \backslash \vec{\ell}_\mathcal{U}, \eta)} \left(|\{\gamma' \in L(\mathcal{U}, \eta) : \gamma'|_{\mathcal{U} \backslash \vec{\ell}_\mathcal{U}} = \gamma\}| - 1 \right)$$

we conclude that

$$P_\eta(\mathcal{U}) - P_\eta(\mathcal{U} \backslash \vec{\ell}_\mathcal{U}) \geq \sum_{\gamma \in L_{\vec{\ell}}(\mathcal{U} \backslash \vec{\ell}_\mathcal{U}, x)} \left(N_{\vec{\ell}, \mathcal{U}}(\gamma) - 1 \right) \geq \left| L_{\vec{\ell}}(\mathcal{U} \backslash \vec{\ell}_\mathcal{U}, x) \right| .$$

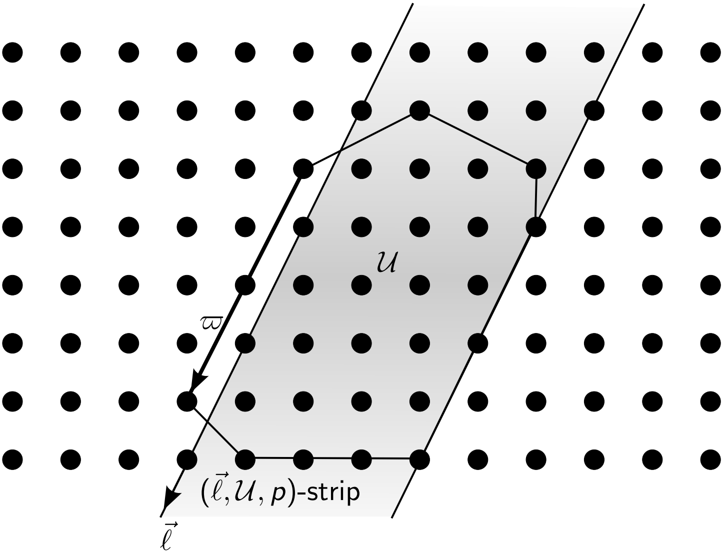
The following lemma shows how nonexpansive directions are connected to the periodicity of some specific configurations.

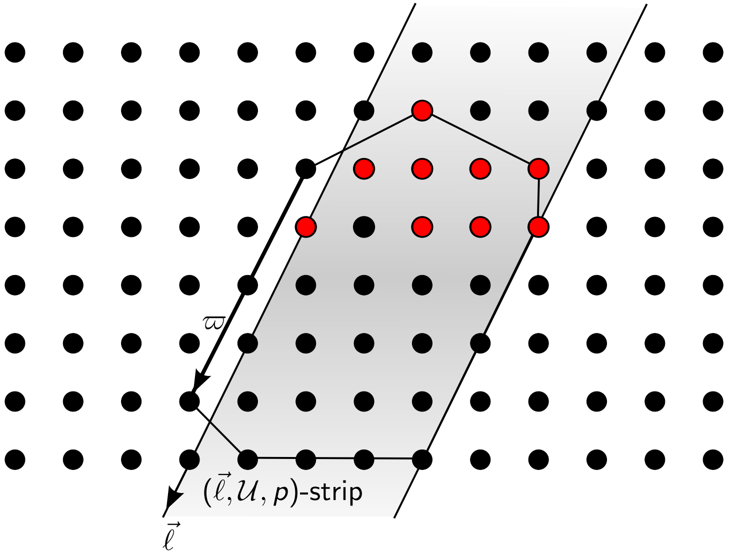
Lemma

Given $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, let $\vec{\ell}$ be a nonexpansive direction on X_η and let $\mathcal{U} \subset \mathbb{Z}^2$ be an η-generating set. For $x \in \mathcal{N}(\vec{\ell}, \mathcal{U})$, if

$$P_\eta(\mathcal{U}) - P_\eta(\mathcal{U} \backslash \vec{\ell}_{\mathcal{U}}) \leq p + |\mathcal{A}(\vec{\ell}, \mathcal{U}, x)| - 2 \quad \text{for some } p \in \mathbb{N},$$

then the restriction of x to the $(\vec{\ell}, \mathcal{U}, p)$-strip is periodic of period tv with $t \leq p + |\mathcal{A}(\vec{\ell}, \mathcal{U}, x)| - 2$.


$\mathcal{U}$
ϖ
$(\vec{\ell}, \mathcal{U}, p)$-strip
$\vec{\ell}$


ϖ
$(\vec{\ell}, \mathcal{U}, p)$-strip
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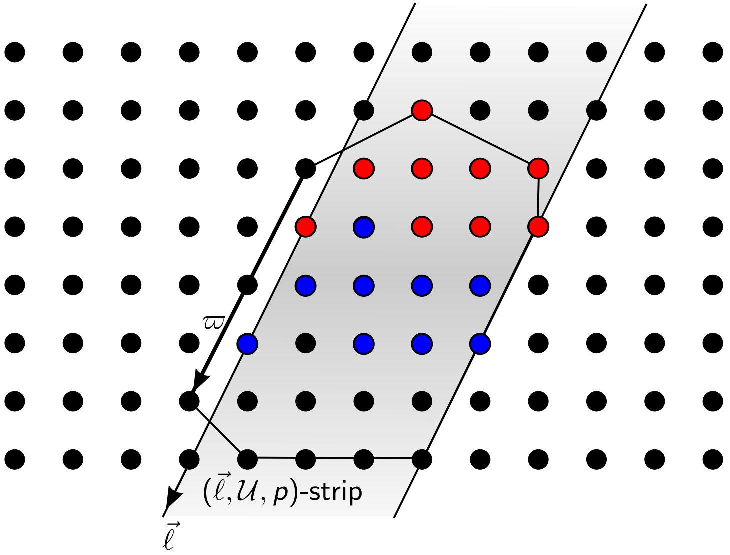
ϖ
$(\vec{\ell}, \mathcal{U}, p)$-strip
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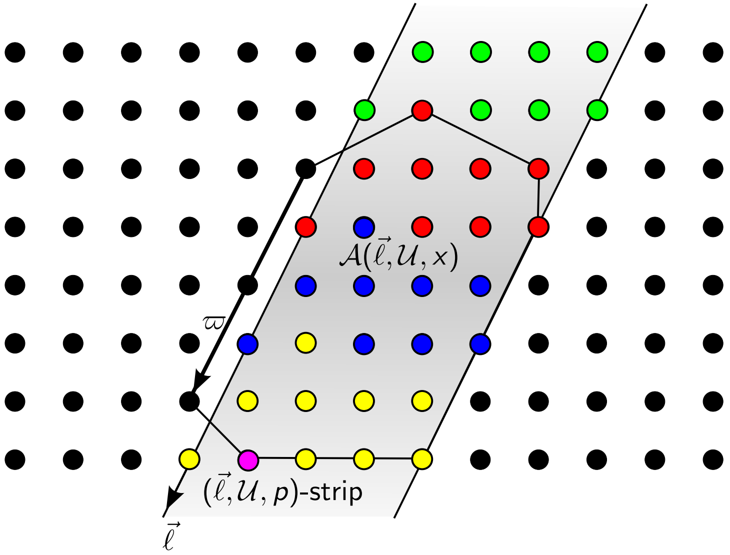
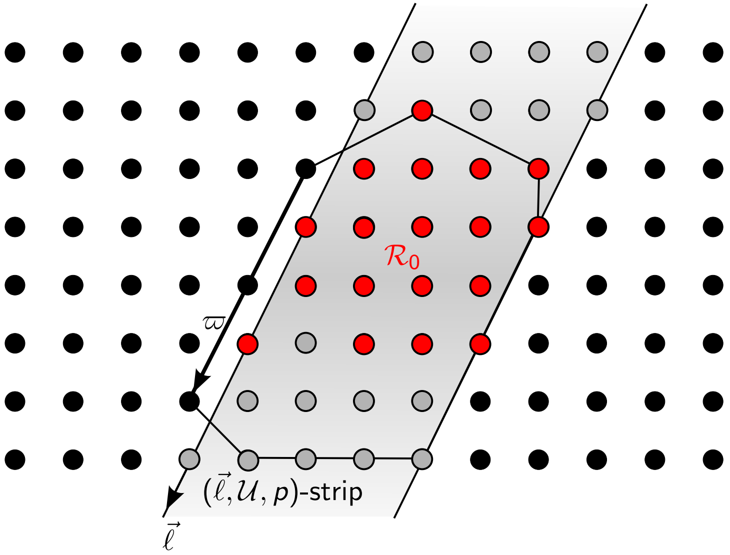
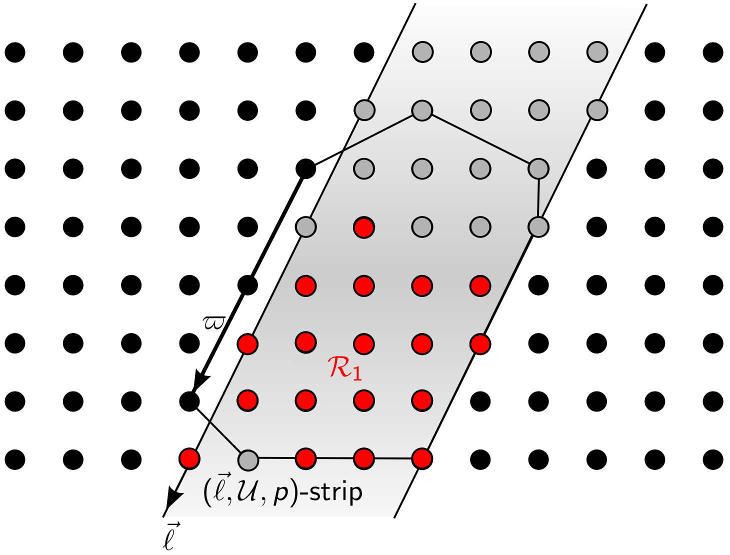

$\mathcal{A}(\vec{\ell},\mathcal{U},x)$
ϖ
$(\vec{\ell},\mathcal{U},p)$-strip
$\vec{\ell}$


$\mathcal{R}_0$
ϖ
$(\vec{\ell}, \mathcal{U}, p)$-strip
$\vec{\ell}$


ϖ
$\mathcal{R}_1$
$(\vec{\ell}, \mathcal{U}, p)$-strip
$\vec{\ell}$

Since $\vec{\ell}$ is a nonexpansive direction, it follows that

$$\left|\{x|_{\mathcal{R}_j} : j \in \mathbb{Z}\}\right| \leq P_\eta(\mathcal{U}) - P_\eta(\mathcal{U}\backslash\varpi) \leq p + |\mathcal{A}(\vec{\ell}, \mathcal{U}, x)| - 2.$$

Hence, the Alphabetical Morse-Hedlund's Theorem implies that the restriction of x to the $(\vec{\ell}, \mathcal{U}, p)$-strip is periodic of period tv with

$$t \leq p + |\mathcal{A}(\vec{\ell}, \mathcal{U}, x)| - 2.$$

Definition

Given $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, let $\vec{\ell} \in \mathbb{G}_1$ be a non-expansive direction on X_η. We say that an $(\eta, \vec{\ell})$-generating set $\mathcal{U} \in \mathcal{F}_{\mathcal{C}}^{Vol}$ is $(\vec{\ell}, p)$-balanced if

(i) for any oriented line $\vec{\ell'} \subset \mathbb{R}^2$ parallel to $\vec{\ell}$, if $\vec{\ell'} \neq \vec{\ell}_{\mathcal{U}}$ and $\ell' \cap \mathcal{U} \neq \emptyset$, then $|\ell' \cap \mathcal{U}| \geq p$,

(ii) for each $x \in \mathcal{N}(\vec{\ell}, \mathcal{U})$ with $|\mathcal{A}(\vec{\ell}, \mathcal{U}, x)| > 1$, there is an integer $p_x \leq p$ such that

$$P_\eta(\mathcal{U}) \leq P_\eta(\mathcal{U} \backslash \vec{\ell}_{\mathcal{U}}) + p_x + |\mathcal{A}(\vec{\ell}, \mathcal{U}, x)| - 2.$$

Definition

A finite, nonempty, convex set $\mathcal{U} \subset \mathbb{Z}^2$ is said to be a quasi-regular set if, for every edge of $\mathcal{U}$, there is an antiparallel edge of $\mathcal{U}$ with the same cardinality.

- If there is a quasi-regular set $\mathcal{U} \in \mathcal{F}_{\mathcal{C}}^{Vol}$ such that $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |\mathcal{A}| - 1$, then for every oriented line $\vec{\ell} \in \mathbb{G}_1$ that is a non-expansive direction on X_η there exists $(\vec{\ell}, p)$-balanced set.
- If $\vec{\ell} \in \mathbb{G}_1$ is a non-expansive direction on X_η and there exists $(\vec{\ell}, p)$-balanced set, then the oriented line $\overleftarrow{\ell} \in \mathbb{G}_1$ antiparallel to $\vec{\ell}$ also is a non-expansive direction on X_η.

Definition

A finite, nonempty, convex set $\mathcal{U} \subset \mathbb{Z}^2$ is said to be a quasi-regular set if, for every edge of $\mathcal{U}$, there is an antiparallel edge of $\mathcal{U}$ with the same cardinality.

- If there is a quasi-regular set $\mathcal{U} \in \mathcal{F}_{\mathcal{C}}^{Vol}$ such that $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |\mathcal{A}| - 1$, then for every oriented line $\vec{\ell} \in \mathbb{G}_1$ that is a non-expansive direction on X_η there exists $(\vec{\ell}, p)$-balanced set.
- If $\vec{\ell} \in \mathbb{G}_1$ is a non-expansive direction on X_η and there exists $(\vec{\ell}, p)$-balanced set, then the oriented line $\overleftarrow{\ell} \in \mathbb{G}_1$ antiparallel to $\vec{\ell}$ also is a non-expansive direction on X_η.

Theorem (Colle's PhD thesis)

For $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ making use of all colors of $\mathcal{A}$, if there is a quasi-regular set $\mathcal{U} \subset \mathbb{Z}^2$ such that $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |\mathcal{A}| - 1$, then η is periodic.

Idea of the proof (following Cyr and Kra)

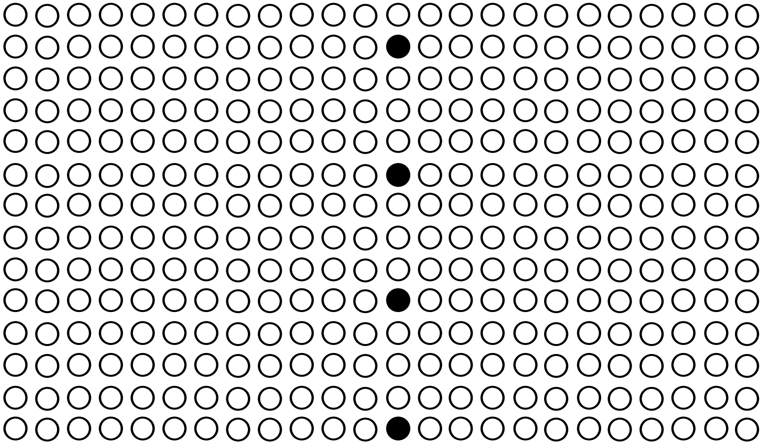
The proof of this theorem is done by contradiction via multiple steps. Assuming the existence of a counter-example, another one is constructed with more structure, i.e., with an unbounded convex region of double periodicity. Fixed a generating set with more properties, a contradiction will arise from the fact that the number of configurations (coloring) occurring at the boundary of this doubly periodic region is greater than possible.

Given $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, suppose that there exist $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq nk$. If $|\mathcal{A}| \geq \frac{1}{2}nk + 1$, then

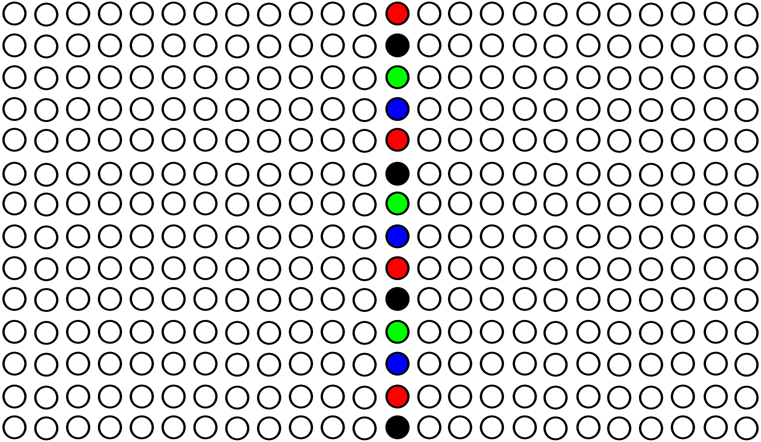
$$P_\eta(n,k) \leq nk \leq \frac{1}{2}nk + |A| - 1,$$

what implies that η is periodic.

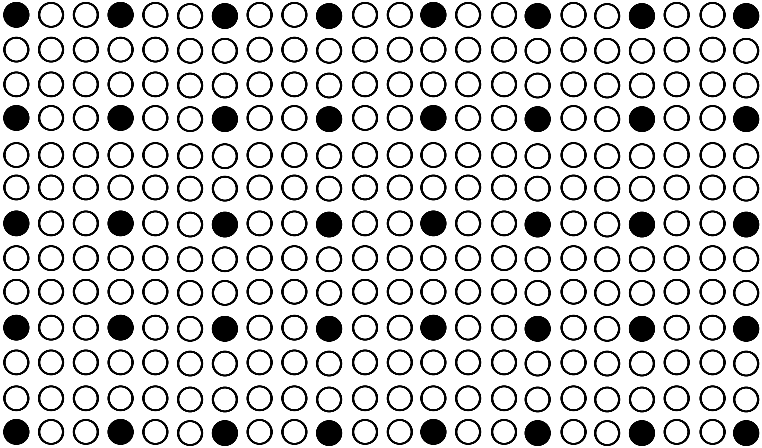
Problem. *Given $\eta \in \mathcal{A}^{\mathbb{Z}^2}$, suppose there exist $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq nk$. Furthermore, suppose $P_\eta(\kappa,\tau) > \kappa\tau$ for every $\kappa, \tau \in \mathbb{N}$ with $1 \leq \kappa\tau < nk$. In this case, there is an alphabet $\mathcal{A}'$ and a configuration $\eta' \in (\mathcal{A}')^{\mathbb{Z}^2}$ such that $|\mathcal{A}'| \geq \frac{1}{2}nk + 1$ and $P_{\eta'}(n,k) = P_\eta(n,k)$?*



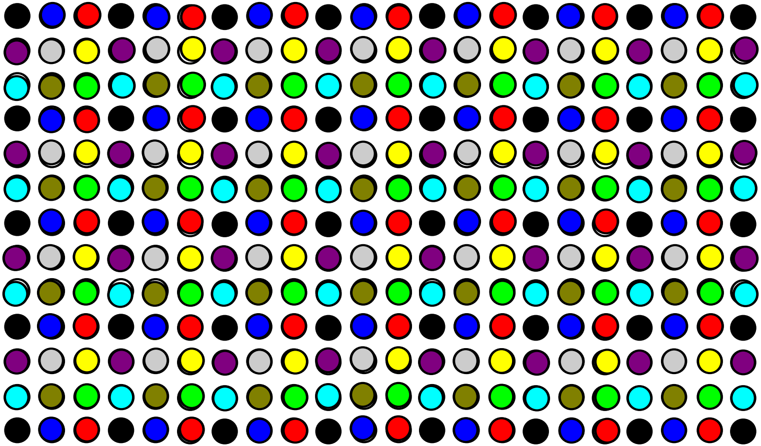
$$|\mathcal{A}| = 2, \quad P_\eta(R_{1,5}) = 5$$



$$|\mathcal{A}'| = 5, \quad P_{\eta'}(R_{1,5}) = 5$$



$$|\mathcal{A}| = 2, \quad P_\eta(R_{3,3}) = 9$$



$$|\mathcal{A}'| = 9, \quad P_{\eta'}(R_{3,3}) = 9$$

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Thank you for your attention!