Reversing and Extended Symmetry groups of shifts

Michael Baake, John Roberts & Reem Yassawi

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Definition (Symmetry group, Reversing symmetry group)

$$\mathcal{S}(\mathbb{X}) := \{\Phi : \mathbb{X} \to \mathbb{X} : \Phi \text{ is a homeomorphism}, \Phi \circ \sigma = \sigma \circ \Phi\}$$

$$\mathcal{R}(\mathbb{X}) := \{\Phi : \mathbb{X} \to \mathbb{X} : \Phi \text{ is a homeomorphism}, \Phi \circ \sigma = \sigma^{\pm 1} \circ \Phi\}$$

The Curtis-Hedlund-Lyndon theorem for Reversors

Lemma

Let $\mathbb{X}$ be a faithful one-dimensional shift over the finite alphabet $\mathcal{A}$. For any reversor $\Phi \in \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X})$, there are non-negative integers ℓ, r and a map $\phi : \mathcal{A}^{\ell+r+1} \to \mathcal{A}$ such that $(\Phi(x))_n = \phi(x_{-n-r}, \dots, x_{-n}, \dots, x_{-n+\ell})$.

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In other words, if $\Phi \in \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X})$, then $\Phi = \Psi \circ R$ where $\Psi : R(X) \to X$ is a sliding block code.

Higher dimensional "reversors"

Let $(\mathbb{X}, \sigma_1, \sigma_2)$ be a $\mathbb{Z}^2$-shift over a finite $\mathcal{A}$ ($\mathcal{G} := \langle \sigma_1, \sigma_2 \rangle \simeq \mathbb{Z}^2$).

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$$\mathcal{S}(\mathbb{X}) = \mathrm{cent}_{\mathcal{H}}(\mathcal{G}) = \{h \in \mathcal{H} : h\sigma_1^m \sigma_2^n = \sigma_1^m \sigma_2^n h, \text{ for } m, n \in \mathbb{Z}\};$$

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Note that $h\mathcal{G} = \mathcal{G}h$ is only possible when the conjugation action $g \mapsto hgh^{-1}$ sends a set of generators of $\mathcal{G}$ to a (possibly different) set of generators. Since $\mathcal{G}$ is a free Abelian group of rank 2 by our assumption, its automorphism group is Aut $(\mathcal{G}) \simeq \mathrm{GL}(2, \mathbb{Z})$, the group of integer 2×2-matrices M with $\det(M) \in \{\pm 1\}$.

Example: The full shift

The extended symmetry group of the full shift $\mathbb{X} = \mathcal{A}^{\mathbb{Z}^d}$ is given by

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$$h_M(x)_{\mathbf{n}} := x_{M^{-1}\mathbf{n}}.$$

Clearly, h_M is a continuous mapping of $\mathbb{X}$ into itself and is invertible.

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Any $h \in \mathcal{R}(\mathbb{X})$ acts on $\sigma_1 \ldots \sigma_d$, this induces $\psi : \mathcal{R}(\mathbb{X}) \to \mathrm{GL}(d,\mathbb{Z}) \simeq \mathcal{K}$.

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So $\varphi \circ \psi$ is a group endomorphism of $\mathcal{R}(\mathbb{X})$ into $\mathcal{K}$ with kernel $\mathcal{S}(\mathbb{X})$, and which fixes $\mathcal{K}$. Hence

$$\mathcal{R}(\mathbb{X}) = \mathcal{S}(\mathbb{X}) \rtimes \mathcal{K} \simeq \mathcal{S}(\mathbb{X}) \rtimes \mathrm{GL}(d, \mathbb{Z}).$$

Some useful lemmas

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Lemma (CHL, extended symmetries)

Let $(\mathbb{X}, \mathbb{Z}^d)$ be a shift over a finite alphabet $\mathcal{A}$, with faithful shift action. Then any $\Phi \in \mathcal{R}(\mathbb{X})$ is of the form $\Phi = \tilde{\Phi} h_M$ with $M = \psi(\Phi)$ and where $\tilde{\Phi} : h_M(\mathbb{X}) \to \mathbb{X}$ is again a sliding block map.

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Lemma (extended symmetry groups as semidirect products)

Let $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a shift with faithful $\mathbb{Z}^d$-action and symmetry group $\mathcal{S}(\mathbb{X})$. Assume further that $\mathcal{R}(\mathbb{X})$ contains a subgroup $\mathcal{H}$ that satisfies $\mathcal{H} \simeq \psi(\mathcal{H})$ together with $\psi(\mathcal{H}) = \psi(\mathcal{R}(\mathbb{X}))$. Then, the extended symmetry group of $(\mathbb{X}, \mathbb{Z}^d)$ is

$$\mathcal{R}(\mathbb{X}) = \mathcal{S}(\mathbb{X}) \rtimes \mathcal{H}.$$

The Chair Substitution's extended symmetry group

Theorem (Baake, Roberts, Y, 2016.)

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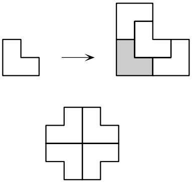

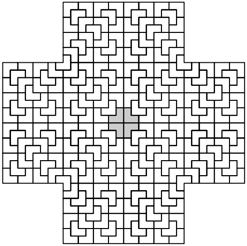

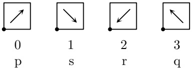

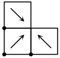

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Lemma (Coven, 1973, Baake-Roberts-Y, 2016)

Let $(\mathbb{X}, \mathbb{Z}^d)$ *be a one-dimensional faithful shift and at least one dense orbit. Suppose further that the group rotation* $(\mathcal{G}, +\alpha)$ *is its MEF, with* $\pi : \mathbb{X} \to \mathcal{G}$ *the corresponding factor map.*
Then, there is a group homomorphism $\kappa : \mathcal{S}(\mathbb{X}) \to \mathcal{G}$ *such that*

$$\pi(\Phi(x)) = \kappa(\Phi) + \pi(x)$$

holds for all $x \in \mathbb{X}$ *and* $G \in \mathcal{S}(\mathbb{X})$.

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holds for all $x \in \mathbb{X}$ and $G \in \mathcal{S}(\mathbb{X})$. Moreover, if $\kappa(\mathbb{Z}^d)$ is a free Abelian group there is an extension of κ to a 1-cocycle of the action of $\mathcal{R}(\mathbb{X})$ on $\mathcal{G}$ by $\zeta : \mathcal{R}(\mathbb{X}) \to \mathcal{H}(\mathcal{G})$, with

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for all $\Phi, \Psi \in \mathcal{R}(\mathbb{X})$. Any $\Phi \in \mathcal{R}(\mathbb{X})$ induces a unique mapping on $\mathcal{G}$ that acts as $z \mapsto \kappa(\Phi) + \zeta(\Phi)(z)$. If $c := \min\{|\pi^{-1}(z)| : z \in \mathcal{G}\} < \infty$, then $\kappa : \mathcal{S}(\mathbb{X}) \to \mathcal{G}$ and $\kappa : \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X}) \to \mathcal{G}$ are each at most c-to-one,

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The Chair substitution's maximal equicontinuous factor

Theorem (Robinson, 1999)

The maximal equicontinuous factor of the Chair shift (X_c, T_1, T_2) *is* $(\mathbb{Z}_2 \times \mathbb{Z}_2, +(1,0), +(0,1))$. *The factor mapping* $\pi : X_c \to \mathbb{Z}_2 \times \mathbb{Z}_2$ *is almost everywhere one-to-one. It is otherwise two-to-one or five-to-one.*

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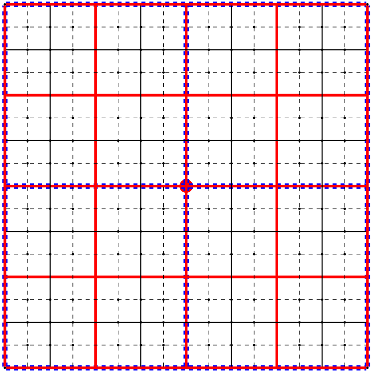
The block structure does not tell you much, or possibly anything, about the entries of the point $x = (x_{m,n})_{m,n \in \mathbb{Z}}$, but only about the way x is tiled with substitution squares.

A block structure

Dashed lines are the basic configuration grid.
Black lines are boundaries of θ-words.
Red lines are boundaries of θ^2-words.
Blue lines are boundaries of θ^3-words.



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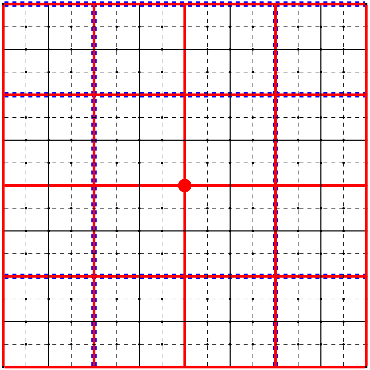

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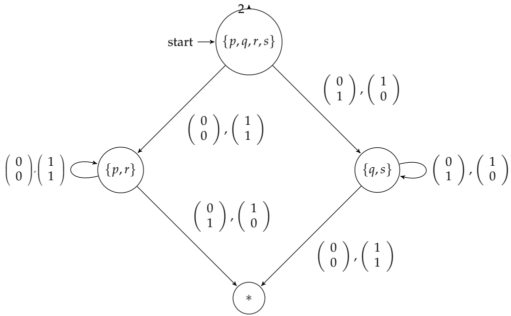

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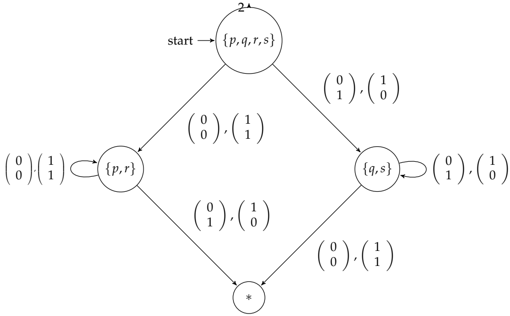
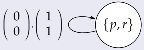
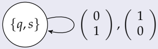


Theorem

The set of complete block structures with multiple preimages are paths that eventually lie in: 



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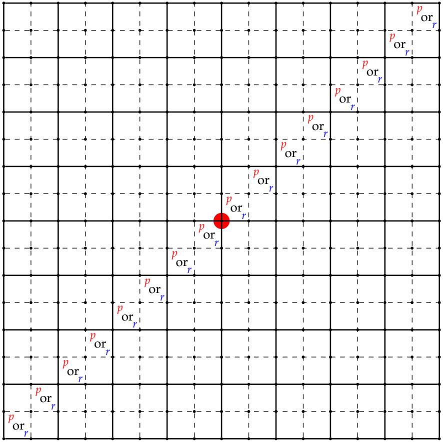

The block structure of a point in $\pi^{-1}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$

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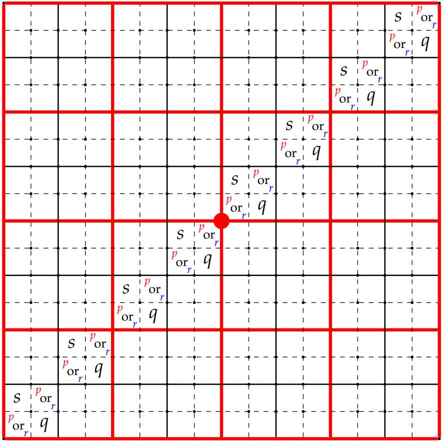

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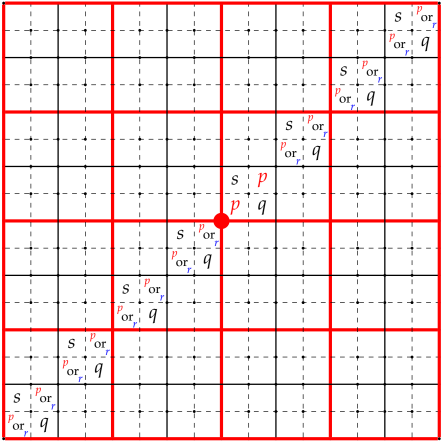

The block structure of a point in $\pi^{-1}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$, with one choice made $x_{0,0} = p$

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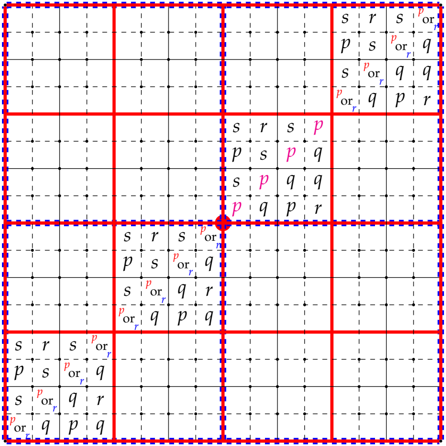
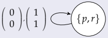
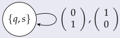
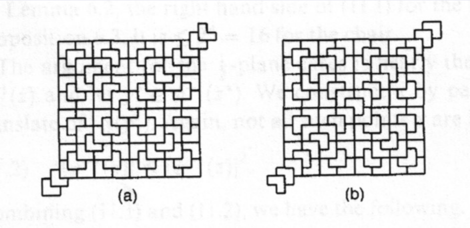


The block structure of a point in $\pi^{-1}\left(\left(\begin{array}{c}0\\0\end{array}\right),\left(\begin{array}{c}0\\0\end{array}\right),\left(\begin{array}{c}0\\0\end{array}\right)\right)$, with $x_{0,0} = p$

In this way we have

Theorem (Robinson, 1999)

If we see a block structure z in   *which has infinitely many occurences of both* $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *then* $|\pi^{-1}(z)| = 2$*, and the two preimages of z agree everywhere off* $y = x$*.*



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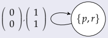
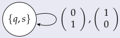
If we see a block structure z in 



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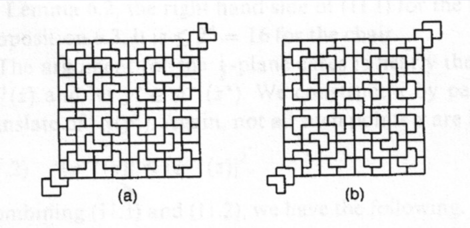

If we see a block structure z which has only entries from $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *then* $|\pi^{-1}(z)| = 2$*, and the two preimages of z agree everywhere off* $y = -x$.

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Theorem (Baake, Roberts, Y, 2016.)

The Chair shift has $\mathbb{Z}^2 \rtimes D_4$ *as extended symmetry group.*

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Recall that we can write every extended symmetry $\Phi = \Psi h_M$ for some $M \in \mathrm{GL}(2\mathbb{Z})$, and Ψ a sliding block code.

The problem with parabolic and hyperbolic matrices

Lemma (Robinson, 1999, Baake-Roberts-Y, 2016)

There exist (uncountably) many pairs of points in $\mathbb{X}_C$ either which

1. *disagree on the main diagonal, and agree everywhere else, or*
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Finishing the proof of the theorem

If M is parabolic or hyperbolic, then

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$$X_c \stackrel{h_M}{\to} h_M(X_c) \stackrel{\Psi}{\to} X_c.$$

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But since Ψ is a sliding block code, $\Psi(h_M(x, y))$ cannot belong to $\mathcal{F}$.

The Ledrappier shift

Some linear cellular automata satisfy symmetry rigidity. Let $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ and let $\mathbb{Y} \subset \mathcal{A}^{\mathbb{Z}^2}$ be the space of all possible spacetime diagrams for Φ; $\mathbb{Y}$ is a 2-dimensional shift $(\mathbb{Y}, \sigma_h, \sigma_v)$, invariant under Haar measure.

Theorem (Kitchens & Schmidt, 2000)

Let Φ_1 and Φ_2 generate shifts of spacetime diagrams $\mathbb{Y}_1$ and $\mathbb{Y}_2$. Suppose that $(\mathbb{Y}_1, \sigma_h, \sigma_v)$ and $(\mathbb{Y}_2, \sigma_h, \sigma_v)$ are mixing and irreducible. Then every measurable conjugacy F is almost everywhere an affine map, i.e. $F(x) = G(x) + c$ almost everywhere with G a group isomorphism.

Theorem (Kitchens & Schmidt, 2000)

If $\Phi : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ is the Ledrappier map $\Phi(x) = x + \sigma(x)$, then any topological self conjugacy $F : \mathbb{Y} \to \mathbb{Y}$ is of the form $F(x) = \sigma_h^m \sigma_v^n(x)$ for some m and n.

The extended symmetries of the Ledrappier shift

Theorem (Baake, Roberts, Y, 2016)

The Ledrappier shift $\mathbb{X}_L$ *has* $\mathbb{Z}^2 \rtimes D_3$ *as extended symmetry group.*

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Some proof ideas: The following matrices send "Ledrappier" triangles to "Ledrappier" triangles

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\};$$

they constitute D_3. The maximal finite subgroup containing D_3 is D_6, which is generated by the extra matrix $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix} \right)^{\frac{1}{2}} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right)$. We will show that M does not give any extended symmetries.

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$$\mathbb{Y} := h_M(\mathbb{X}_L) = \{x \in \{0,1\}^{\mathbb{Z}^2} : x_{(m,n)} + x_{(m+1,n)} + x_{(m+1,n-1)} \equiv 0 \mod 2\}.$$

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Note that $\mathbb{Y} \neq \mathbb{X}_L$ so if there is a reversor, it must be of the form $\Phi = \Psi \circ h_M$, with $\Psi : \mathbb{Y} \to \mathbb{X}_L$ nontrivial.

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Now to exclude elements of infinite order, refine this argument, along with use of

Lemma

The only annihilating triangles in X_L of area $\frac{1}{2}$ are the Ledrappier triangles.