Reversing and Extended Symmetry groups of shifts

Michael Baake, John Roberts & Reem Yassawi

Friday 16th June 2017
Definition

Let \((X, \sigma)\) be a faithful shift action. A homeomorphism \(\Phi : X \to X\) is a symmetry of \((X, \sigma)\) if \(\Phi \circ \sigma = \sigma \circ \Phi\).

Example

\(X = \mathbb{A}\mathbb{Z}, R(x)_n := x - n; R\) is a reflection.

Definition (Symmetry group, Reversing symmetry group)

\[ S(X) := \{ \Phi : X \to X : \Phi \text{ is a homeomorphism, } \Phi \circ \sigma = \sigma \circ \Phi \} \]

\[ R(X) := \{ \Phi : X \to X : \Phi \text{ is a homeomorphism, } \Phi \circ \sigma = \sigma^{-1} \circ \Phi \} \]
Definition

Let \((X, \sigma)\) be a faithful shift action. A homeomorphism \(\Phi : X \to X\) is a **symmetry** of \((X, \sigma)\) if \(\Phi \circ \sigma = \sigma \circ \Phi\).

Our symmetry=your automorphism!

Example

\(X = \text{A\ Z, R}\) : \(x_n := x - n\); \(R\) is a reflection.

Definition (Symmetry group, Reversing symmetry group)

\[ S(X) := \{ \Phi : X \to X : \Phi \text{ is a homeomorphism, } \Phi \circ \sigma = \sigma \circ \Phi \} \]

\[ R(X) := \{ \Phi : X \to X : \Phi \text{ is a homeomorphism, } \Phi \circ \sigma = \sigma \pm 1 \circ \Phi \} \]
Definition

Let \((X, \sigma)\) be a faithful shift action. A homeomorphism \(\Phi : X \to X\) is a symmetry of \((X, \sigma)\) if \(\Phi \circ \sigma = \sigma \circ \Phi\).

Our symmetry=your automorphism!
Our automorphism=your homeomorphism!
Definition

Let \((X, \sigma)\) be a faithful shift action. A homeomorphism \(\Phi : X \to X\) is a symmetry of \((X, \sigma)\) if \(\Phi \circ \sigma = \sigma \circ \Phi\).

Our symmetry=your automorphism!
Our automorphism=your homeomorphism!

Definition

Let \((X, \sigma)\) be a faithful shift action. A homeomorphism \(\Phi : X \to X\) is a reversor of \((X, \sigma)\) if \(\Phi \circ \sigma = \sigma^{-1} \circ \Phi\).
Definition

Let $(X, \sigma)$ be a faithful shift action. A homeomorphism $\Phi : X \to X$ is a symmetry of $(X, \sigma)$ if $\Phi \circ \sigma = \sigma \circ \Phi$.

Our symmetry=your automorphism!
Our automorphism=your homeomorphism!

Definition

Let $(X, \sigma)$ be a faithful shift action. A homeomorphism $\Phi : X \to X$ is a reversor of $(X, \sigma)$ if $\Phi \circ \sigma = \sigma^{-1} \circ \Phi$.

Example

$X = \mathcal{A}^\mathbb{Z}$, $R(x)_n := x_{-n}$; $R$ is a reflection.
Definition

Let \((X, \sigma)\) be a faithful shift action. A homeomorphism \(\Phi : X \to X\) is a symmetry of \((X, \sigma)\) if \(\Phi \circ \sigma = \sigma \circ \Phi\).

Our symmetry=your automorphism!
Our automorphism=your homeomorphism!

Definition

Let \((X, \sigma)\) be a faithful shift action. A homeomorphism \(\Phi : X \to X\) is a reversor of \((X, \sigma)\) if \(\Phi \circ \sigma = \sigma^{-1} \circ \Phi\).

Example

\(X = \mathcal{A}^\mathbb{Z},\ R(x)_n := x_{-n};\ R\) is a reflection.

Definition (Symmetry group, Reversing symmetry group)

\(S(X) := \{\Phi : X \to X : \Phi\) is a homeomorphism,\(\Phi \circ \sigma = \sigma \circ \Phi\}\)
\(R(X) := \{\Phi : X \to X : \Phi\) is a homeomorphism,\(\Phi \circ \sigma = \sigma^{\pm1} \circ \Phi\}\)
Lemma

Let $X$ be a faithful one-dimensional shift over the finite alphabet $A$. For any reversor $\Phi \in R(X) \setminus S(X)$, there are non-negative integers $\ell, r$ and a map $\phi : A^{\ell+r+1} \to A$ such that $(\Phi(x))_n = \phi(x_{-n-r}, \ldots, x_{-n}, \ldots, x_{-n+\ell})$. In other words, if $\Phi \in R(X) \setminus S(X)$, then $\Phi = \Psi \circ R$ where $\Psi : R(X) \to X$ is a sliding block code.
Lemma

Let $X$ be a faithful one-dimensional shift over the finite alphabet $A$. For any reversor $\Phi \in R(X) \setminus S(X)$, there are non-negative integers $\ell, r$ and a map $\phi : A^{\ell+r+1} \to A$ such that $(\Phi(x))_n = \phi(x_{-n-r}, \ldots, x_{-n}, \ldots, x_{-n+\ell})$.

In other words, if $\Phi \in R(X) \setminus S(X)$, then $\Phi = \Psi \circ R$ where $\Psi : R(X) \to X$ is a sliding block code.
Higher dimensional "reversors"

Let \((X, \sigma_1, \sigma_2)\) be a \(\mathbb{Z}^2\)-shift over a finite \(\mathcal{A} \ (\mathcal{G} := \langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}^2)\).
Higher dimensional "reversors"

Let \((X, \sigma_1, \sigma_2)\) be a \(\mathbb{Z}^2\)-shift over a finite \(\mathcal{A} (\mathcal{G} := \langle \sigma_1, \sigma_2 \rangle \simeq \mathbb{Z}^2)\). Let \(\mathcal{H}\) be the group of homeomorphisms of \(X\). Then

\[
\mathcal{S}(X) = \text{cent}_{\mathcal{H}}(\mathcal{G}) = \{ h \in \mathcal{H} : h\sigma_1^m\sigma_2^n = \sigma_1^m\sigma_2^n h, \text{ for } m, n \in \mathbb{Z} \};
\]
Let \((X, \sigma_1, \sigma_2)\) be a \(\mathbb{Z}^2\)-shift over a finite \(A (G := \langle \sigma_1, \sigma_2 \rangle \simeq \mathbb{Z}^2)\). Let \(\mathcal{H}\) be the group of homeomorphisms of \(X\). Then

\[
\mathcal{S}(X) = \text{cent}_{\mathcal{H}}(G) = \{ h \in \mathcal{H} : h\sigma_1^m\sigma_2^n = \sigma_1^m\sigma_2^n h, \text{ for } m, n \in \mathbb{Z} \};
\]

so we define the extended symmetry group

\[
\mathcal{R}(X) := \text{norm}_{\mathcal{H}}(G) = \{ h \in \mathcal{H} : hG = G h \}.
\]
Let \((X, \sigma_1, \sigma_2)\) be a \(\mathbb{Z}^2\)-shift over a finite \(A\) \((\mathcal{G} : = \langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}^2)\). Let \(\mathcal{H}\) be the group of homeomorphisms of \(X\). Then

\[ S(X) = \text{cent}_{\mathcal{H}}(\mathcal{G}) = \{ h \in \mathcal{H} : h\sigma_1^m\sigma_2^n = \sigma_1^m\sigma_2^n h, \text{ for } m, n \in \mathbb{Z} \}; \]

so we define the extended symmetry group

\[ \mathcal{R}(X) : = \text{norm}_{\mathcal{H}}(\mathcal{G}) = \{ h \in \mathcal{H} : h\mathcal{G} = \mathcal{G} h \}. \]

Note that \(h\mathcal{G} = \mathcal{G} h\) is only possible when the conjugation action \(g \mapsto hgh^{-1}\) sends a set of generators of \(\mathcal{G}\) to a (possibly different) set of generators. Since \(\mathcal{G}\) is a free Abelian group of rank 2 by our assumption, its automorphism group is \(\text{Aut}(\mathcal{G}) \cong \text{GL}(2, \mathbb{Z})\), the group of integer \(2 \times 2\)-matrices \(M\) with \(\det(M) \in \{\pm 1\}\).
Example: The full shift

The extended symmetry group of the full shift $X = \mathcal{A}^{\mathbb{Z}^d}$ is given by

$$\mathcal{R}(X) \cong \mathcal{S}(X) \rtimes \text{GL}(d, \mathbb{Z}).$$
Example: The full shift

The extended symmetry group of the full shift $\mathbb{X} = A^{\mathbb{Z}^d}$ is given by

$$\mathcal{R}(\mathbb{X}) \simeq S(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z}).$$

Let $M \in \text{GL}(d, \mathbb{Z})$ and consider the mapping $x \mapsto h_M(x)$ defined by

$$h_M(x)_n := x_{M^{-1}n}.$$ 

Clearly, $h_M$ is a continuous mapping of $\mathbb{X}$ into itself and is invertible.
Example: The full shift

The extended symmetry group of the full shift $\mathbb{X} = \mathcal{A}^{\mathbb{Z}^d}$ is given by

$$\mathcal{R}(\mathbb{X}) \simeq \mathcal{S}(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z}).$$

Let $M \in \text{GL}(d, \mathbb{Z})$ and consider the mapping $x \mapsto h_M(x)$ defined by

$$h_M(x)_n := x_{M^{-1}n}.$$

Clearly, $h_M$ is a continuous mapping of $\mathbb{X}$ into itself and is invertible. Also, $h_M h_{M'} = h_{MM'}$, whence $\varphi : \text{GL}(d, \mathbb{Z}) \rightarrow \mathcal{H}(\mathbb{X})$ defined by $M \mapsto h_M$ is a group homomorphism.
Example: The full shift

The extended symmetry group of the full shift $\mathbb{X} = \mathcal{A}^\mathbb{Z}$ is given by

$$\mathcal{R}(\mathbb{X}) \simeq S(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z}).$$

Let $M \in \text{GL}(d, \mathbb{Z})$ and consider the mapping $x \mapsto h_M(x)$ defined by

$$h_M(x)_n := x_{M^{-1}n}.$$

Clearly, $h_M$ is a continuous mapping of $\mathbb{X}$ into itself and is invertible. Also, $h_M h_{M'} = h_{MM'}$, whence $\varphi : \text{GL}(d, \mathbb{Z}) \rightarrow \mathcal{H}(\mathbb{X})$ defined by $M \mapsto h_M$ is a group homomorphism.

Given $M$, the action $g \mapsto h_M g h_M^{-1}$ on $\mathcal{G}$ sends $\sigma_i$ to $\prod_{j} \sigma_j^{m_{ji}}$, for any $1 \leq i \leq d$, so $\varphi(M) \in \mathcal{R}(\mathbb{X})$. 
Example: The full shift

The extended symmetry group of the full shift $\mathbb{X} = \mathcal{A}^\mathbb{Z}^d$ is given by

$$\mathcal{R}(\mathbb{X}) \cong \mathcal{S}(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z}).$$

Let $M \in \text{GL}(d, \mathbb{Z})$ and consider the mapping $x \mapsto h_M(x)$ defined by

$$h_M(x)_n := x_{M^{-1}n}.$$

Clearly, $h_M$ is a continuous mapping of $\mathbb{X}$ into itself and is invertible. Also, $h_M h_{M'} = h_{MM'}$, whence $\varphi : \text{GL}(d, \mathbb{Z}) \to \mathcal{H}(\mathbb{X})$ defined by $M \mapsto h_M$ is a group homomorphism.

Given $M$, the action $g \mapsto h_M g h_M^{-1}$ on $\mathcal{G}$ sends $\sigma_i$ to $\prod_j \sigma_j^{m_{ji}}$, for any $1 \leq i \leq d$, so $\varphi(M) \in \mathcal{R}(\mathbb{X})$.

Consider $\mathcal{K} := \varphi(\text{GL}(d, \mathbb{Z})) \leq \mathcal{R}(\mathbb{X})$. Since $\varphi$ is injective, $\mathcal{K} \cong \text{GL}(d, \mathbb{Z})$. 

Reem Yassawi
Reversing and Extended Symmetry group
Friday 16th June 2017
The extended symmetry group of the full shift $\mathbb{X} = \mathcal{A}^{\mathbb{Z}^d}$ is given by

$$\mathcal{R}(\mathbb{X}) \cong \mathcal{S}(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z}).$$

Let $M \in \text{GL}(d, \mathbb{Z})$ and consider the mapping $x \mapsto h_M(x)$ defined by

$$h_M(x)_n := x_{M^{-1}n}.$$

Clearly, $h_M$ is a continuous mapping of $\mathbb{X}$ into itself and is invertible. Also, $h_M h_{M'} = h_{MM'}$, whence $\varphi : \text{GL}(d, \mathbb{Z}) \to \mathcal{H}(\mathbb{X})$ defined by $M \mapsto h_M$ is a group homomorphism.

Given $M$, the action $g \mapsto h_M gh_M^{-1}$ on $\mathcal{G}$ sends $\sigma_i$ to $\prod_j \sigma_j^{m_{ji}}$, for any $1 \leq i \leq d$, so $\varphi(M) \in \mathcal{R}(\mathbb{X})$.

Consider $\mathcal{K} := \varphi(\text{GL}(d, \mathbb{Z})) \leq \mathcal{R}(\mathbb{X})$. Since $\varphi$ is injective, $\mathcal{K} \cong \text{GL}(d, \mathbb{Z})$.

Any $h \in \mathcal{R}(\mathbb{X})$ acts on $\sigma_1 \ldots \sigma_d$, this induces $\psi : \mathcal{R}(\mathbb{X}) \to \text{GL}(d, \mathbb{Z}) \cong \mathcal{K}$. 
Example: The full shift

The extended symmetry group of the full shift $\mathbb{X} = A^{\mathbb{Z}^d}$ is given by

$$\mathcal{R}(\mathbb{X}) \cong S(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z}).$$

Let $M \in \text{GL}(d, \mathbb{Z})$ and consider the mapping $x \mapsto h_M(x)$ defined by

$$h_M(x)_n := x_{M^{-1}n}.$$

Clearly, $h_M$ is a continuous mapping of $\mathbb{X}$ into itself and is invertible. Also, $h_Mh_{M'} = h_{MM'}$, whence $\varphi : \text{GL}(d, \mathbb{Z}) \to \mathcal{H}(\mathbb{X})$ defined by $M \mapsto h_M$ is a group homomorphism.

Given $M$, the action $g \mapsto h_Mgh_M^{-1}$ on $G$ sends $\sigma_i$ to $\prod_j \sigma_j^{m_{ji}}$, for any $1 \leq i \leq d$, so $\varphi(M) \in \mathcal{R}(\mathbb{X})$.

Consider $\mathcal{K} := \varphi(\text{GL}(d, \mathbb{Z})) \leq \mathcal{R}(\mathbb{X})$. Since $\varphi$ is injective, $\mathcal{K} \cong \text{GL}(d, \mathbb{Z})$.

Any $h \in \mathcal{R}(\mathbb{X})$ acts on $\sigma_1 \ldots \sigma_d$, this induces $\psi : \mathcal{R}(\mathbb{X}) \to \text{GL}(d, \mathbb{Z}) \cong \mathcal{K}$. So $\varphi \circ \psi$ is a group endomorphism of $\mathcal{R}(\mathbb{X})$ into $\mathcal{K}$ with kernel $S(\mathbb{X})$, and which fixes $\mathcal{K}$. Hence

$$\mathcal{R}(\mathbb{X}) = S(\mathbb{X}) \rtimes \mathcal{K} \cong S(\mathbb{X}) \rtimes \text{GL}(d, \mathbb{Z}).$$
Some useful lemmas

Recall the definition of $\psi : \mathcal{R}(X) \to \text{GL}(d, \mathbb{Z})$. 

Lemma (CHL, extended symmetries)
Let $(X, Z^d)$ be a shift over a finite alphabet $A$, with faithful shift action. Then any $\Phi \in \mathcal{R}(X)$ is of the form $\Phi = \tilde{\Phi} h M$ with $M = \psi(\Phi)$ and where $\tilde{\Phi} : h M(X) \to X$ is again a sliding block map.

Lemma (extended symmetry groups as semidirect products)
Let $X \subseteq A Z^d$ be a shift with faithful $Z^d$-action and symmetry group $S(X)$. Assume further that $\mathcal{R}(X)$ contains a subgroup $H$ that satisfies $H \cong \psi(H)$ together with $\psi(H) = \psi(\mathcal{R}(X))$. Then, the extended symmetry group of $(X, Z^d)$ is $\mathcal{R}(X) = S(X) \rtimes H$. 

Reem Yassawi
Reversing and Extended Symmetry groups
Friday 16th June 2017 6 / 22
Some useful lemmas

Recall the definition of $\psi : \mathcal{R}(X) \to \text{GL}(d, \mathbb{Z})$.

**Lemma (CHL, extended symmetries)**

Let $(X, \mathbb{Z}^d)$ be a shift over a finite alphabet $\mathcal{A}$, with faithful shift action. Then any $\Phi \in \mathcal{R}(X)$ is of the form $\Phi = \tilde{\Phi}h_M$ with $M = \psi(\Phi)$ and where $\tilde{\Phi} : h_M(X) \to X$ is again a sliding block map.
Some useful lemmas

Recall the definition of $\psi : \mathcal{R}(X) \rightarrow \text{GL}(d, \mathbb{Z})$.

**Lemma (CHL, extended symmetries)**

Let $(X, \mathbb{Z}^d)$ be a shift over a finite alphabet $\mathcal{A}$, with faithful shift action. Then any $\Phi \in \mathcal{R}(X)$ is of the form $\Phi = \tilde{\Phi} h_M$ with $M = \psi(\Phi)$ and where $\tilde{\Phi} : h_M(X) \rightarrow X$ is again a sliding block map.

**Lemma (extended symmetry groups as semidirect products)**

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a shift with faithful $\mathbb{Z}^d$-action and symmetry group $S(X)$. Assume further that $\mathcal{R}(X)$ contains a subgroup $\mathcal{H}$ that satisfies $\mathcal{H} \cong \psi(\mathcal{H})$ together with $\psi(\mathcal{H}) = \psi(\mathcal{R}(X))$. Then, the extended symmetry group of $(X, \mathbb{Z}^d)$ is

$$\mathcal{R}(X) = S(X) \rtimes \mathcal{H}.$$
The Chair shift $X_C$ has $\mathbb{Z}^2 \rtimes D_4$ as extended symmetry group.
The Chair Substitution’s extended symmetry group

**Theorem (Baake, Roberts, Y, 2016.)**

The Chair shift $\mathbb{C}$ has $\mathbb{Z}^2 \rtimes D_4$ as extended symmetry group.

Theorem (Baake, Roberts, Y, 2016.)

The Chair shift $X_C$ has $\mathbb{Z}^2 \rtimes D_4$ as extended symmetry group.

\[\begin{array}{c}
\theta(p) = \begin{bmatrix}
s & p \\
p & q \\
\end{bmatrix} \\
\theta(r) = \begin{bmatrix}
s & r \\
r & q \\
\end{bmatrix} \\
\theta(q) = \begin{bmatrix}
q & r \\
p & q \\
\end{bmatrix} \\
\theta(s) = \begin{bmatrix}
s & r \\
p & s \\
\end{bmatrix}
\end{array}\]

Reem Yassawi

Reversing and Extended Symmetry group

Friday 16th June 2017 7 / 22

Let $(X, \mathbb{Z}^d)$ be a one-dimensional faithful shift and at least one dense orbit. Suppose further that the group rotation $(G, +\alpha)$ is its MEF, with $\pi : X \to G$ the corresponding factor map.

Then, there is a group homomorphism $\kappa : S(X) \to G$ such that

$$\pi(\Phi(x)) = \kappa(\Phi) + \pi(x)$$

holds for all $x \in X$ and $G \in S(X)$. 


Let \((X, \mathbb{Z}^d)\) be a one-dimensional faithful shift and at least one dense orbit. Suppose further that the group rotation \((G, +\alpha)\) is its MEF, with \(\pi : X \rightarrow G\) the corresponding factor map.

Then, there is a group homomorphism \(\kappa : S(X) \rightarrow G\) such that

\[
\pi(\Phi(x)) = \kappa(\Phi) + \pi(x)
\]

holds for all \(x \in X\) and \(G \in S(X)\). Moreover, if \(\kappa(\mathbb{Z}^d)\) is a free Abelian group there is an extension of \(\kappa\) to a 1-cocycle of the action of \(\mathcal{R}(X)\) on \(G\) by \(\zeta : \mathcal{R}(X) \rightarrow \mathcal{H}(G)\), with

\[
\kappa(\Phi \Psi) = \kappa(\Phi) + \zeta(\Phi)(\kappa(\Psi))
\]

for all \(\Phi, \Psi \in \mathcal{R}(X)\). Any \(\Phi \in \mathcal{R}(X)\) induces a unique mapping on \(G\) that acts as \(z \mapsto \kappa(\Phi) + \zeta(\Phi)(z)\).

Let \((X, \mathbb{Z}^d)\) be a one-dimensional faithful shift and at least one dense orbit. Suppose further that the group rotation \((G, +\alpha)\) is its MEF, with \(\pi : X \to G\) the corresponding factor map. Then, there is a group homomorphism \(\kappa : \mathcal{S}(X) \to G\) such that

\[
\pi(\Phi(x)) = \kappa(\Phi) + \pi(x)
\]

holds for all \(x \in X\) and \(G \in \mathcal{S}(X)\). Moreover, if \(\kappa(\mathbb{Z}^d)\) is a free Abelian group there is an extension of \(\kappa\) to a 1-cocycle of the action of \(\mathcal{R}(X)\) on \(G\) by \(\zeta : \mathcal{R}(X) \to \mathcal{H}(G)\), with

\[
\kappa(\Phi\Psi) = \kappa(\Phi) + \zeta(\Phi)(\kappa(\Psi))
\]

for all \(\Phi, \Psi \in \mathcal{R}(X)\). Any \(\Phi \in \mathcal{R}(X)\) induces a unique mapping on \(G\) that acts as \(z \mapsto \kappa(\Phi) + \zeta(\Phi)(z)\). If \(c := \min\{|\pi^{-1}(z)| : z \in G\} < \infty\), then \(\kappa : \mathcal{S}(X) \to G\) and \(\kappa : \mathcal{R}(X) \setminus \mathcal{S}(X) \to G\) are each at most \(c\)-to-one,

Let \((X, \mathbb{Z}^d)\) be a one-dimensional faithful shift and at least one dense orbit. Suppose further that the group rotation \((G, +\alpha)\) is its MEF, with \(\pi : X \to G\) the corresponding factor map.

Then, there is a group homomorphism \(\kappa : S(X) \to G\) such that

\[
\pi(\Phi(x)) = \kappa(\Phi) + \pi(x)
\]

holds for all \(x \in X\) and \(G \in S(X)\). Moreover, if \(\kappa(\mathbb{Z}^d)\) is a free Abelian group there is an extension of \(\kappa\) to a 1-cocycle of the action of \(R(X)\) on \(G\) by \(\zeta : R(X) \to H(G)\), with

\[
\kappa(\Phi \Psi) = \kappa(\Phi) + \zeta(\Phi)(\kappa(\Psi))
\]

for all \(\Phi, \Psi \in R(X)\). Any \(\Phi \in R(X)\) induces a unique mapping on \(G\) that acts as \(z \mapsto \kappa(\Phi) + \zeta(\Phi)(z)\). If \(c := \min\{|\pi^{-1}(z)| : z \in G\} < \infty\), then \(\kappa : S(X) \to G\) and \(\kappa : R(X) \setminus S(X) \to G\) are each at most \(c\)-to-one, and for any \(d > c\), \(\{z \in G : |\pi^{-1}(z)| = d\} = \zeta(\{z \in G : |\pi^{-1}(z)| = d\}) + \kappa(\Phi)\) holds for each \(\Phi \in R(X)\).
The Chair substitution’s maximal equicontinuous factor

**Theorem (Robinson, 1999)**

The maximal equicontinuous factor of the Chair shift \((X_c, T_1, T_2)\) is 
\((\mathbb{Z}_2 \times \mathbb{Z}_2, +(1, 0), +(0, 1))\). The factor mapping \(\pi : X_c \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2\) is almost everywhere one-to-one. It is otherwise two-to-one or five-to-one.

"The points where the factor mapping fails to be 1-to-1 correspond to interesting tilings".

Each point in \(X_c\) is associated with a block structure, which is described by a point in \(\mathbb{Z}_2 \times \mathbb{Z}_2\). The block structure does not tell you much, or possibly anything, about the entries of the point \(x = (x_m, n)\) \(m, n \in \mathbb{Z}\), but only about the way \(x\) is tiled with substitution squares.
The Chair substitution’s maximal equicontinuous factor

Theorem (Robinson, 1999)

The maximal equicontinuous factor of the Chair shift \((X_c, T_1, T_2)\) is \((\mathbb{Z}_2 \times \mathbb{Z}_2, +(1, 0), +(0, 1))\). The factor mapping \(\pi : X_c \to \mathbb{Z}_2 \times \mathbb{Z}_2\) is almost everywhere one-to-one. It is otherwise two-to-one or five-to-one.

"The points where the factor mapping fails to be 1-to-1 correspond to interesting tilings".
The Chair substitution’s maximal equicontinuous factor

Theorem (Robinson, 1999)

The maximal equicontinuous factor of the Chair shift \((X_c, T_1, T_2)\) is \((\mathbb{Z}_2 \times \mathbb{Z}_2, +(1, 0), +(0, 1))\). The factor mapping \(\pi: X_c \to \mathbb{Z}_2 \times \mathbb{Z}_2\) is almost everywhere one-to-one. It is otherwise two-to-one or five-to-one.

"The points where the factor mapping fails to be 1-to-1 correspond to interesting tilings". Each point in \(X_c\) is associated with a block structure, which is described by a point in \(\mathbb{Z}_2 \times \mathbb{Z}_2\).
The Chair substitution’s maximal equicontinuous factor

Theorem (Robinson, 1999)

The maximal equicontinuous factor of the Chair shift \((X_c, T_1, T_2)\) is \((\mathbb{Z}_2 \times \mathbb{Z}_2, +(1, 0), +(0, 1))\). The factor mapping \(\pi : X_c \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2\) is almost everywhere one-to-one. It is otherwise two-to-one or five-to-one.

"The points where the factor mapping fails to be 1-to-1 correspond to interesting tilings". Each point in \(X_c\) is associated with a block structure, which is described by a point in \(\mathbb{Z}_2 \times \mathbb{Z}_2\).

The block structure does not tell you much, or possibly anything, about the entries of the point \(x = (x_m,n)_{m,n \in \mathbb{Z}}\), but only about the way \(x\) is tiled with substitution squares.
Dashed lines are the basic configuration grid. Black lines are boundaries of $\theta$-words. Red lines are boundaries of $\theta^2$-words. Blue lines are boundaries of $\theta^3$-words.
Another block structure

Dashed lines are the basic configuration grid.
Black lines are boundaries of $\theta$-words.
Red lines are boundaries of $\theta^2$-words.
Blue lines are boundaries of $\theta^3$-words.

The block structure of a point in $\pi^{-1}\left(\left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right)$
The block structure graph: Finding repeated block structures

\[
\theta(p) = \begin{array}{cc}
  s & p \\
p & q \\
\end{array} \quad \theta(r) = \begin{array}{cc}
  s & r \\
r & q \\
\end{array} \quad \theta(q) = \begin{array}{cc}
  q & r \\
p & q \\
\end{array} \quad \theta(s) = \begin{array}{cc}
  s & r \\
p & s \\
\end{array}
\]
The block structure graph: Finding repeated block structures

\[ \theta(p) = \begin{bmatrix} s & p \\ p & q \end{bmatrix} \]
\[ \theta(r) = \begin{bmatrix} s & r \\ r & q \end{bmatrix} \]
\[ \theta(q) = \begin{bmatrix} q & r \\ p & q \end{bmatrix} \]
\[ \theta(s) = \begin{bmatrix} s & r \\ p & s \end{bmatrix} \]

Theorem
The set of complete block structures with multiple preimages are paths that eventually lie in:
\[ \{ p, q, r, s \} \]
\[ \{ p, r \} \]
\[ \{ q, s \} \]
\[ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]

Reem Yassawi
Reversing and Extended Symmetry group
Friday 16th June 2017 12 / 22
The block structure graph: Finding repeated block structures

\[ \theta(p) = \begin{bmatrix} s & p \\ p & q \end{bmatrix}, \quad \theta(r) = \begin{bmatrix} s & r \\ r & q \end{bmatrix}, \quad \theta(q) = \begin{bmatrix} q & r \\ p & q \end{bmatrix}, \quad \theta(s) = \begin{bmatrix} s & r \\ p & s \end{bmatrix} \]

Theorem

The set of complete block structures with multiple preimages are paths that eventually lie in:

\[ \{p, r\} \quad \{q, s\} \]

\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
The Chair substitution

\[ \theta(p) = \begin{array}{cc}
  s & p \\
  p & q \\
\end{array} \quad \quad \theta(r) = \begin{array}{cc}
  s & r \\
  r & q \\
\end{array} \quad \quad \theta(q) = \begin{array}{cc}
  q & r \\
  p & q \\
\end{array} \quad \quad \theta(s) = \begin{array}{cc}
  s & r \\
  p & s \\
\end{array} \]

The block structure of a point in \( \pi^{-1}(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) \)
The Chair substitution

\[ \theta(p) = \begin{array}{cc} s & p \\ p & q \end{array} \quad \theta(r) = \begin{array}{cc} s & r \\ r & q \end{array} \quad \theta(q) = \begin{array}{cc} q & r \\ p & q \end{array} \quad \theta(s) = \begin{array}{cc} s & r \\ p & s \end{array}\]

The block structure of a point in \(\pi^{-1}\left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right)\)
The Chair substitution

\[
\begin{array}{cc}
\theta(p) &=& \begin{array}{cc}
s & p \\
p & q \\
\end{array} \\
\theta(r) &=& \begin{array}{cc}
s & r \\
r & q \\
\end{array} \\
\theta(q) &=& \begin{array}{cc}
q & r \\
p & q \\
\end{array} \\
\theta(s) &=& \begin{array}{cc}
q & r \\
p & q \\
\end{array} \\
\end{array}
\]

The block structure of a point in \(\pi^{-1} \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \right)\), with one choice made \(x_{0,0} = p\)
The Chair substitution

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The block structure of a point in $\pi^{-1}\left(\left(\frac{0}{0}\right), \left(\frac{0}{0}\right), \left(\frac{0}{0}\right)\right)$, with $x_{0,0} = p$
In this way we have

**Theorem (Robinson, 1999)**

*If we see a block structure \( z \) in \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix} \) which has infinitely many occurrences of both \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) then \( |\pi^{-1}(z)| = 2 \), and the two preimages of \( z \) agree everywhere off \( y = x \).*
In this way we have

**Theorem (Robinson, 1999)**

If we see a block structure $z$ in $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ which has infinitely many occurrences of both $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $|\pi^{-1}(z)| = 2$, and the two preimages of $z$ agree everywhere off $y = x$.

If we see a block structure $z$ which has only entries from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $|\pi^{-1}(z)| = 2$, and the two preimages of $z$ agree everywhere off $y = -x$. 
The Chair substitution

Theorem (Baake, Roberts, Y, 2016.)

The Chair shift has $\mathbb{Z}^2 \rtimes D_4$ as extended symmetry group.

Sketch of Proof

"The" $\sigma$-fixed point has square symmetry, which is why every element of $D_4$ corresponds to an extended symmetry. $D_4$ is a maximal finite subgroup of $\text{GL}(2, \mathbb{Z})$. Thus, by Selberg's lemma, if there are any other extended symmetries they must by generated by a matrix with infinite order: i.e. a parabolic or hyperbolic matrix. We need to show there are no such extended symmetries. Recall that we can write every extended symmetry $\Phi = \Psi \circ h$ for some $M \in \text{GL}(2, \mathbb{Z})$, and $\Psi$ a sliding block code.
The Chair substitution

Theorem (Baake, Roberts, Y, 2016.)

The Chair shift has $\mathbb{Z}^2 \rtimes D_4$ as extended symmetry group.

Sketch of Proof
"The" $\sigma$-fixed point has square symmetry, which is why every element of $D_4$ corresponds to an extended symmetry.
Theorem (Baake, Roberts, Y, 2016.)

The Chair shift has $\mathbb{Z}^2 \rtimes D_4$ as extended symmetry group.

Sketch of Proof

"The" $\sigma$-fixed point has square symmetry, which is why every element of $D_4$ corresponds to an extended symmetry. $D_4$ is a maximal finite subgroup of $GL(2, \mathbb{Z})$. 

$D_4$ is a maximal finite subgroup of $GL(2, \mathbb{Z})$. Thus, by Selberg's lemma, if there are any other extended symmetries they must by generated by a matrix with infinite order: i.e. a parabolic or hyperbolic matrix. We need to show there are no such extended symmetries. Recall that we can write every extended symmetry $\Phi = \Psi M$ for some $M \in GL(2, \mathbb{Z})$, and $\Psi$ a sliding block code.
The Chair substitution

**Theorem (Baake, Roberts, Y, 2016.)**

*The Chair shift has $\mathbb{Z}^2 \rtimes D_4$ as extended symmetry group.*

**Sketch of Proof**

"The" $\sigma$-fixed point has square symmetry, which is why every element of $D_4$ corresponds to an extended symmetry. $D_4$ is a maximal finite subgroup of $GL(2, \mathbb{Z})$. Thus, by Selberg’s lemma, if there are any other extended symmetries they must by generated by a matrix with infinite order: i.e. a parabolic or hyperbolic matrix.
The Chair substitution

Theorem (Baake, Roberts, Y, 2016.)

The Chair shift has \( \mathbb{Z}^2 \rtimes D_4 \) as extended symmetry group.

Sketch of Proof

"The" \( \sigma \)-fixed point has square symmetry, which is why every element of \( D_4 \) corresponds to an extended symmetry. \( D_4 \) is a maximal finite subgroup of \( GL(2, \mathbb{Z}) \).

Thus, by Selberg’s lemma, if there are any other extended symmetries they must by generated by a matrix with infinite order: i.e. a parabolic or hyperbolic matrix.

We need to show there are no such extended symmetries.
The Chair substitution

Theorem (Baake, Roberts, Y, 2016.)

The Chair shift has \( \mathbb{Z}^2 \rtimes D_4 \) as extended symmetry group.

Sketch of Proof

"The" \( \sigma \)-fixed point has square symmetry, which is why every element of \( D_4 \) corresponds to an extended symmetry. 

\( D_4 \) is a maximal finite subgroup of \( GL(2, \mathbb{Z}) \). 

Thus, by Selberg’s lemma, if there are any other extended symmetries they must be generated by a matrix with infinite order: i.e. a parabolic or hyperbolic matrix. 

We need to show there are no such extended symmetries. 

Recall that we can write every extended symmetry \( \Phi = \Psi h_M \) for some \( M \in GL(2\mathbb{Z}) \), and \( \Psi \) a sliding block code.
Lemma (Robinson, 1999, Baake-Roberts-Y, 2016)

There exist (uncountably) many pairs of points in $\mathbb{X}_C$ either which

1. disagree on the main diagonal, and agree everywhere else, or

2. disagree on the diagonal $y = -x$ and agree everywhere else.
Lemma (Robinson, 1999, Baake-Roberts-Y, 2016)

There exist (uncountably) many pairs of points in $X_C$ either which

1. disagree on the main diagonal, and agree everywhere else, or
2. disagree on the diagonal $y = -x$ and agree everywhere else.

Let $\mathcal{F}$ be the set of pairs of such points. An extended symmetry must send an element of $\mathcal{F}$ to an element of $\mathcal{F}$.
The problem with parabolic and hyperbolic matrices

**Lemma (Robinson, 1999, Baake-Roberts-Y, 2016)**

There exist (uncountably) many pairs of points in \( \mathbb{X}_C \) either which

1. disagree on the main diagonal, and agree everywhere else, or
2. disagree on the diagonal \( y = -x \) and agree everywhere else.

Let \( \mathcal{F} \) be the set of pairs of such points. An extended symmetry must send an element of \( \mathcal{F} \) to an element of \( \mathcal{F} \).

**Finishing the proof of the theorem**

If \( M \) is parabolic or hyperbolic, then

\[
M(\{y = x, \text{ or } y = -x\}) \neq \{y = x, y = -x\}.
\]
The problem with parabolic and hyperbolic matrices

Lemma (Robinson, 1999, Baake-Roberts-Y, 2016)

There exist (uncountably) many pairs of points in $X_C$ either which

1. disagree on the main diagonal, and agree everywhere else, or
2. disagree on the diagonal $y = -x$ and agree everywhere else.

Let $\mathcal{F}$ be the set of pairs of such points. An extended symmetry must send an element of $\mathcal{F}$ to an element of $\mathcal{F}$.

Finishing the proof of the theorem

If $M$ is parabolic or hyperbolic, then

$$M(\{y = x, \text{ or } y = -x\}) \neq \{y = x, y = -x\}.$$

Thus $h_M$ sends an element of $\mathcal{F}$ to a pair $(x, y)$ not in $\mathcal{F}$. 
The problem with parabolic and hyperbolic matrices

**Lemma (Robinson, 1999, Baake-Roberts-Y, 2016)**

There exist (uncountably) many pairs of points in $X_C$ either which

1. disagree on the main diagonal, and agree everywhere else, or
2. disagree on the diagonal $y = -x$ and agree everywhere else.

Let $F$ be the set of pairs of such points. An extended symmetry must send an element of $F$ to an element of $F$.

**Finishing the proof of the theorem**

If $M$ is parabolic or hyperbolic, then

$$M(\{y = x, \text{ or } y = -x\}) \neq \{y = x, y = -x\}.$$ 

Thus $h_M$ sends an element of $F$ to a pair $(x, y)$ not in $F$. But $\Phi = \Psi \circ h_M : X_c \rightarrow X_c$ sends elements of $F$ to points in $F$.

$$X_c \xrightarrow{h_M} h_M(X_c) \xrightarrow{\psi} X_c.$$
The problem with parabolic and hyperbolic matrices

Lemma (Robinson, 1999, Baake-Roberts-Y, 2016)

There exist (uncountably) many pairs of points in $X_C$ either which

1. disagree on the main diagonal, and agree everywhere else, or
2. disagree on the diagonal $y = -x$ and agree everywhere else.

Let $\mathcal{F}$ be the set of pairs of such points. An extended symmetry must send an element of $\mathcal{F}$ to an element of $\mathcal{F}$.

Finishing the proof of the theorem

If $M$ is parabolic or hyperbolic, then

$$M(\{y = x, \text{ or } y = -x\}) \neq \{y = x, y = -x\}.$$ 

Thus $h_M$ sends an element of $\mathcal{F}$ to a pair $(x, y)$ not in $\mathcal{F}$. But $\Phi = \Psi \circ h_M : X_c \to X_c$ sends elements of $\mathcal{F}$ to points in $\mathcal{F}$.

$$X_c \xrightarrow{h_M} h_M(X_c) \xrightarrow{\Psi} X_c.$$ 

But since $\Psi$ is a sliding block code, $\Psi(h_M(x, y))$ cannot belong to $\mathcal{F}$. 

Reem Yassawi

Reversing and Extended Symmetry group  
Friday 16th June 2017 19 / 22
The Ledrappier shift

Some linear cellular automata satisfy symmetry rigidity. Let $\Phi : \mathbb{A}^\mathbb{Z} \to \mathbb{A}^\mathbb{Z}$ and let $\mathbb{Y} \subset \mathbb{A}^{\mathbb{Z}^2}$ be the space of all possible spacetime diagrams for $\Phi$; $\mathbb{Y}$ is a 2-dimensional shift $(\mathbb{Y}, \sigma_h, \sigma_v)$, invariant under Haar measure.

**Theorem (Kitchens & Schmidt, 2000)**

Let $\Phi_1$ and $\Phi_2$ generate shifts of spacetime diagrams $\mathbb{Y}_1$ and $\mathbb{Y}_2$. Suppose that $(\mathbb{Y}_1, \sigma_h, \sigma_v)$ and $(\mathbb{Y}_2, \sigma_h, \sigma_v)$ are mixing and irreducible. Then every measurable conjugacy $F$ is almost everywhere an affine map, i.e. $F(x) = G(x) + c$ almost everywhere with $G$ a group isomorphism.

**Theorem (Kitchens & Schmidt, 2000)**

If $\Phi : \{0, 1\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}$ is the Ledrappier map $\Phi(x) = x + \sigma(x)$, then any topological self conjugacy $F : \mathbb{Y} \to \mathbb{Y}$ is of the form $F(x) = \sigma_h^m\sigma_v^n(x)$ for some $m$ and $n$. 
The extended symmetries of the Ledrappier shift

Theorem (Baake, Roberts, Y, 2016)

The Ledrappier shift $X_L$ has $\mathbb{Z}^2 \rtimes D_3$ as extended symmetry group.
The extended symmetries of the Ledrappier shift

Theorem (Baake, Roberts, Y, 2016)

The Ledrappier shift $X_L$ has $\mathbb{Z}^2 \rtimes D_3$ as extended symmetry group.

Some proof ideas: The following matrices send "Ledrappier" triangles to "Ledrappier" triangles

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \right\};$$

they constitute $D_3$. The maximal finite subgroup containing $D_3$ is $D_6$, which is generated by the extra matrix $M = \left( \begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix} \right)^{1/2} = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right)$. We will show that $M$ does not give any extended symmetries.
The extended symmetries of the Ledrappier shift

Theorem (Baake, Roberts, Y, 2016)

The Ledrappier shift $X_L$ has $\mathbb{Z}^2 \rtimes D_3$ as extended symmetry group.

Some proof ideas: The following matrices send "Ledrappier" triangles to "Ledrappier" triangles

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\};$$

they constitute $D_3$. The maximal finite subgroup containing $D_3$ is $D_6$, which is generated by the extra matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. We will show that $M$ does not give any extended symmetries. A computation gives that

$$Y := h_M(X_L) = \{ x \in \{0, 1\}^{\mathbb{Z}^2} : x(m,n) + x(m+1,n) + x(m+1,n-1) \equiv 0 \mod 2 \}. $$
$X_L = \{ x \in \{0, 1\}^\mathbb{Z}^2 : x(m, n) + x(m+1, n) + x(m, n+1) \equiv 0 \mod 2 \}$.

$Y := h_M(X_L) = \{ x \in \{0, 1\}^\mathbb{Z}^2 : x(m, n) + x(m+1, n) + x(m+1, n-1) \equiv 0 \mod 2 \}$.
$$\mathcal{X}_L = \{ x \in \{0, 1\} \mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m,n+1) \equiv 0 \mod 2 \}.$$ 

$$\mathcal{Y} := h_M(\mathcal{X}_L) = \{ x \in \{0, 1\} \mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m+1,n-1) \equiv 0 \mod 2 \}.$$ 

Note that $\mathcal{Y} \neq \mathcal{X}_L$ so if there is a reversor, it must be of the form $\Phi = \Psi \circ h_M$, with $\Psi : \mathcal{Y} \to \mathcal{X}_L$ nontrivial.
\( \mathbb{X}_L = \{ x \in \{0, 1\} \mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m,n+1) \equiv 0 \mod 2 \}. \)

\( \mathbb{Y} := h_M(\mathbb{X}_L) = \{ x \in \{0, 1\} \mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m+1,n-1) \equiv 0 \mod 2 \}. \)

Note that \( \mathbb{Y} \neq \mathbb{X}_L \) so if there is a reversor, it must be of the form \( \Phi = \Psi \circ h_M, \) with \( \Psi : \mathbb{Y} \to \mathbb{X}_L \) nontrivial.

Knowledge of a row in \( x \in \mathbb{X}_L \) determines everything above it.
\[
X_L = \{ x \in \{0, 1\}^{\mathbb{Z}^2} : x(m,n) + x(m+1,n) + x(m,n+1) \equiv 0 \pmod{2} \}.
\]

\[
Y := h_M(X_L) = \{ x \in \{0, 1\}^{\mathbb{Z}^2} : x(m,n) + x(m+1,n) + x(m+1,n-1) \equiv 0 \pmod{2} \}.
\]

Note that \( Y \neq X_L \) so if there is a reversor, it must be of the form \( \Phi = \Psi \circ h_M \), with \( \Psi : Y \rightarrow X_L \) nontrivial.

Knowledge of a row in \( x \in X_L \) determines everything above it.
Knowledge of a row in \( y \in Y \) determines everything below it.
\[ X_L = \{ x \in \{0, 1\}^2 : x(m,n) + x(m+1,n) + x(m,n+1) \equiv 0 \pmod{2} \} . \]

\[ Y := h_M(X_L) = \{ x \in \{0, 1\}^2 : x(m,n) + x(m+1,n) + x(m+1,n-1) \equiv 0 \pmod{2} \} . \]

Note that \( Y \neq X_L \) so if there is a reversor, it must be of the form \( \Phi = \Psi \circ h_M \), with \( \Psi : Y \to X_L \) nontrivial.

Knowledge of a row in \( x \in X_L \) determines everything above it.
Knowledge of a row in \( y \in Y \) determines everything below it.
So the image a row \( r \) in \( x \in X_L \), which is a row \( r^* \) in \( Y \), determines the image of the half plane below \( r^* \). But knowledge of \( r^* \) determines the half plane above \( r^* \). This contradicts the fact that \( \Psi \) is injective.
\[
X_L = \{ x \in \{0, 1\} \mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m,n+1) \equiv 0 \mod 2 \}.
\]

\[
Y := h_M(X_L) = \{ x \in \{0, 1\} \mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m+1,n-1) \equiv 0 \mod 2 \}.
\]

Note that \( Y \neq X_L \) so if there is a reversor, it must be of the form \( \Phi = \Psi \circ h_M \), with \( \Psi : Y \to X_L \) nontrivial.

Knowledge of a row in \( x \in X_L \) determines everything above it.
Knowledge of a row in \( y \in Y \) determines everything below it.
So the image a row \( r \) in \( x \in X_L \), which is a row \( r^* \) in \( Y \), determines the image of the half plane below \( r^* \). But knowledge of \( r^* \) determines the half plane above \( r^* \). This contradicts the fact that \( \Psi \) is injective.

So \( D_3 \) is the maximal finite subgroup in \( \mathcal{R}(X_L) \).
\[ X_L = \{ x \in \{0, 1\}^\mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m,n+1) \equiv 0 \mod 2 \}. \]

\[ Y := h_M(X_L) = \{ x \in \{0, 1\}^\mathbb{Z}^2 : x(m,n) + x(m+1,n) + x(m+1,n-1) \equiv 0 \mod 2 \}. \]

Note that \( Y \neq X_L \) so if there is a reversor, it must be of the form \( \Phi = \Psi \circ h_M \), with \( \Psi : Y \to X_L \) nontrivial.

Knowledge of a row in \( x \in X_L \) determines everything above it.
Knowledge of a row in \( y \in Y \) determines everything below it.
So the image a row \( r \) in \( x \in X_L \), which is a row \( r^* \) in \( Y \), determines the image of the half plane below \( r^* \). But knowledge of \( r^* \) determines the half plane above \( r^* \). This contradicts the fact that \( \Psi \) is injective.
So \( D_3 \) is the maximal finite subgroup in \( \mathcal{R}(X_L) \).
Now to exclude elements of infinite order, refine this argument, along with use of

**Lemma**

*The only annihilating triangles in \( X_L \) of area \( \frac{1}{2} \) are the Ledrappier triangles.*