Determinism in subshifts

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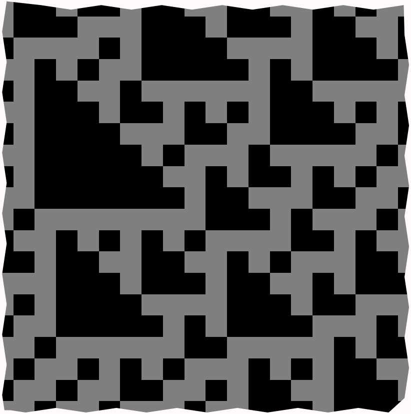
Subshifts

- **alphabet** : finite set $\mathcal{A}$
- **configuration** : $x \in \mathcal{A}^{\mathbb{Z}^d}$
- **pattern** : $x_{|F}$, $F \subset_{\text{finite}} \mathbb{Z}^d$
- **shift** : $\sigma^i(x)_j = x_{i+j}$
- **subshift** :
  closed σ-invariant $\mathcal{X} \subset \mathcal{A}^{\mathbb{Z}^d}$
  $\leftrightarrow \mathcal{X} = \left\{ x \middle| \forall i, F, x_{|F+i} \notin \mathcal{F} \right\}$
  for some $\mathcal{F} \subset \bigcup_{F \subset_{\text{finite}} \mathcal{A}^{\mathbb{Z}^d}} \mathcal{A}^F$
- **SFT** : $\mathcal{X}$ with $\mathcal{F}$ finite
- **effective** : $\mathcal{X}$ with $\mathcal{F}$ computable



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- Let $\mathcal{X}$ a subshift
- $U \subseteq \mathbb{Z}^d$ **codes** $V \subseteq \mathbb{Z}^d$ $(U \underset{\mathcal{X}}{\longrightarrow} V)$:
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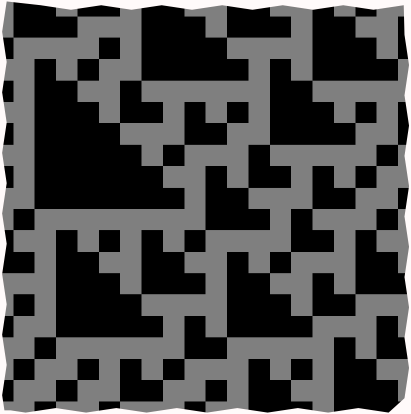

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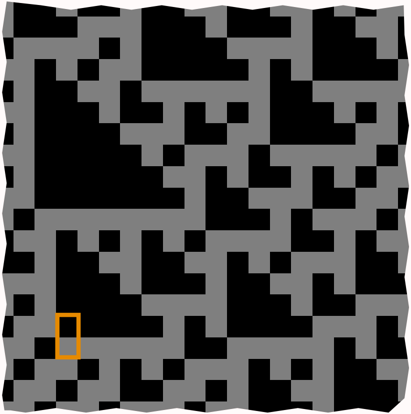

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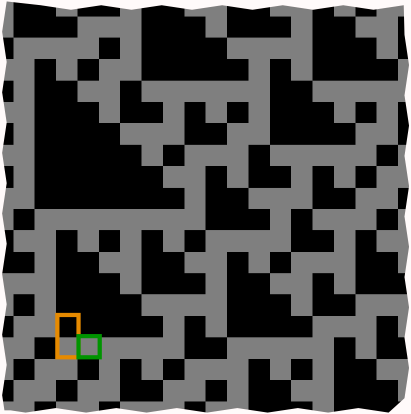

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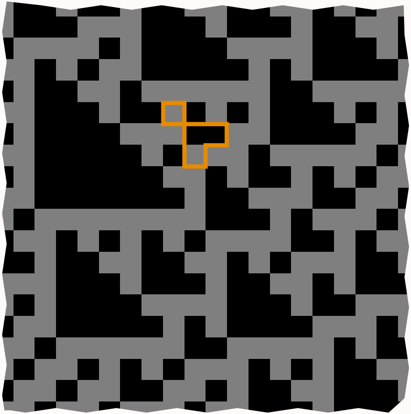

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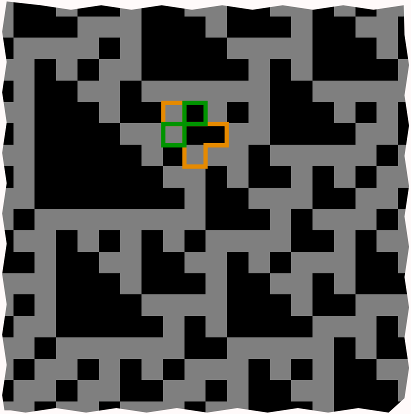

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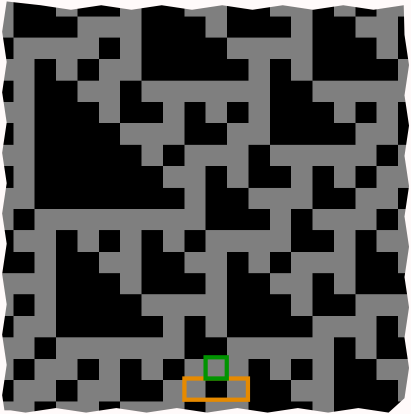

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- e.g. *SW-deterministic* tile set: $\{(0,1),(1,0)\} \underset{\mathcal{X}}{\longrightarrow} \{(0,0)\}$

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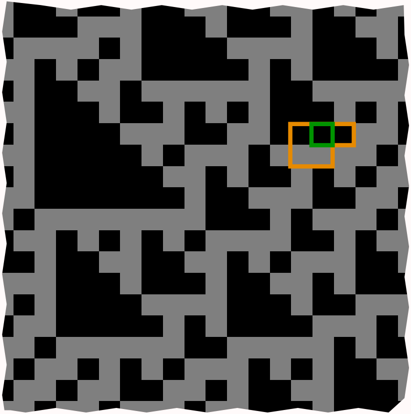

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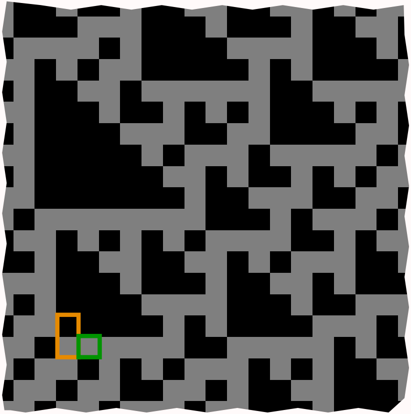

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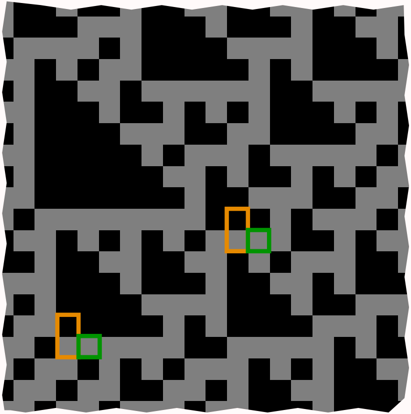

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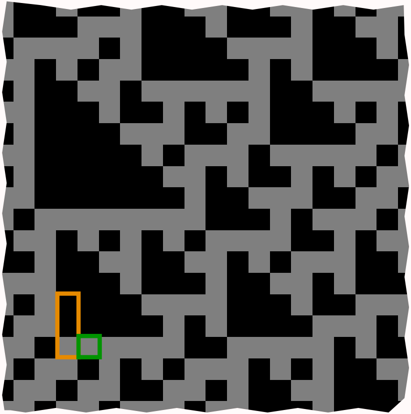

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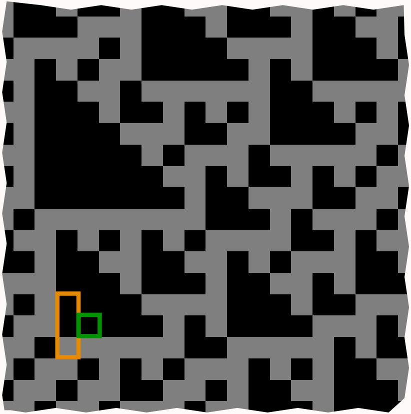

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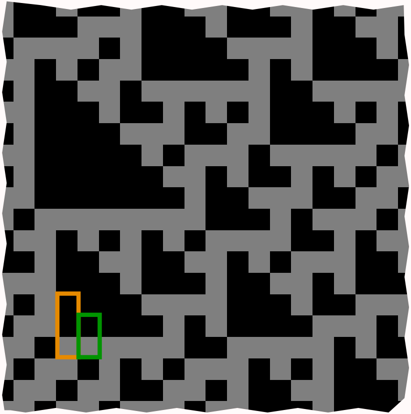

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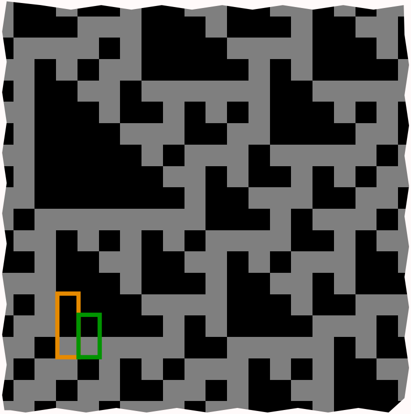

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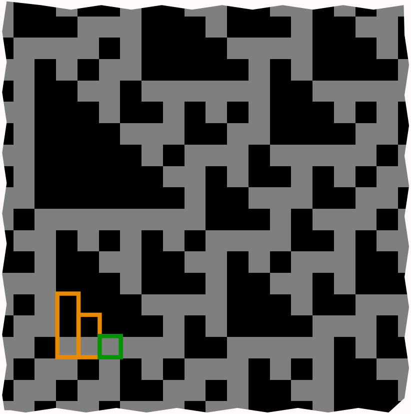

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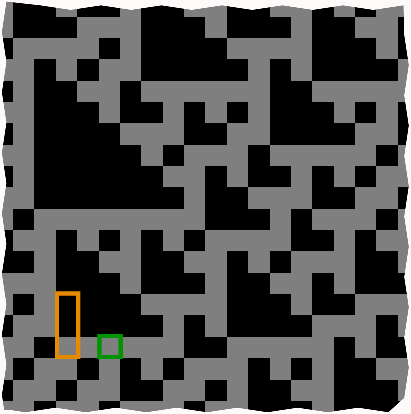

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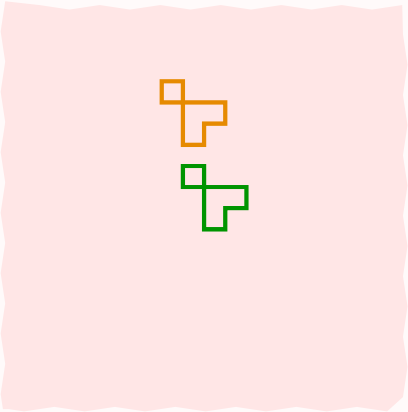

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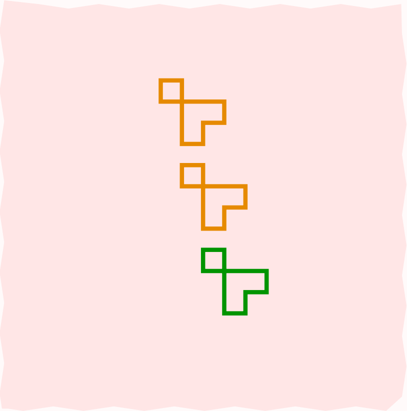

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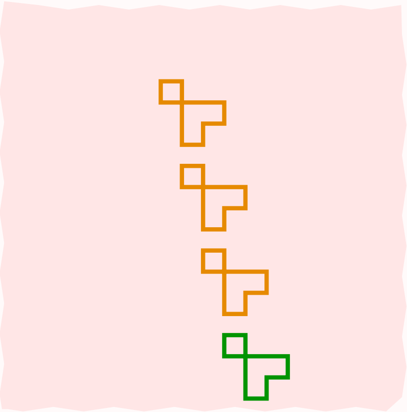

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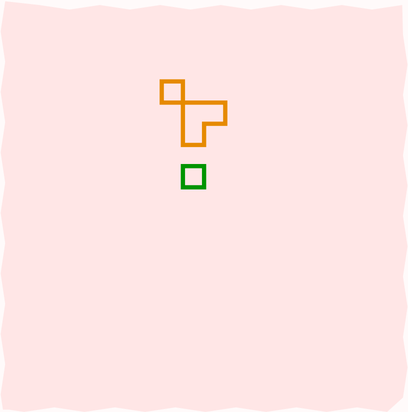

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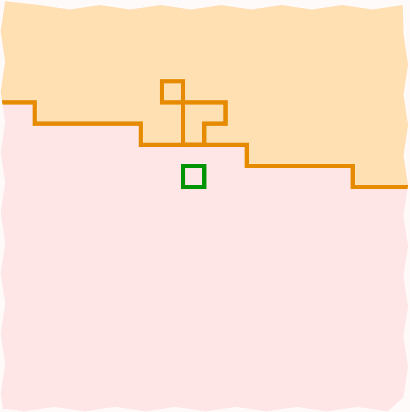

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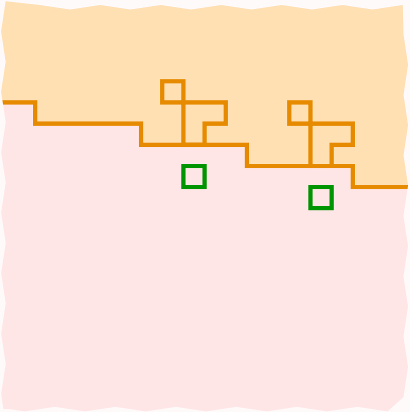

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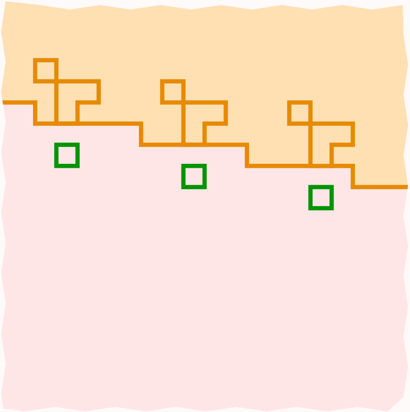

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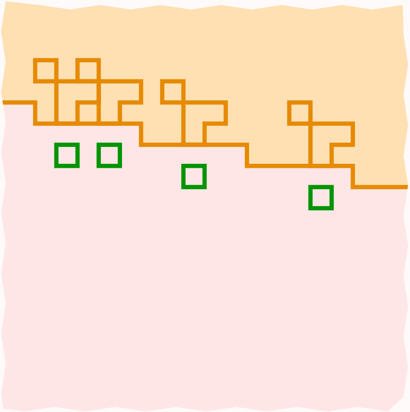

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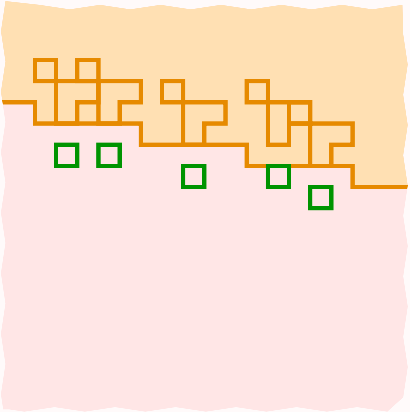

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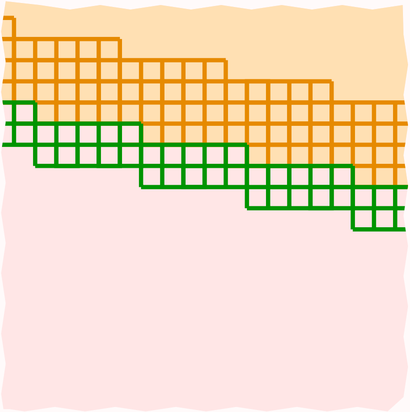

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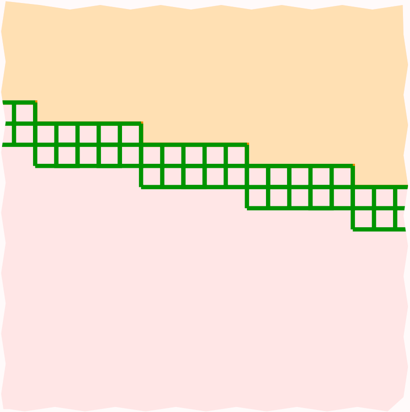

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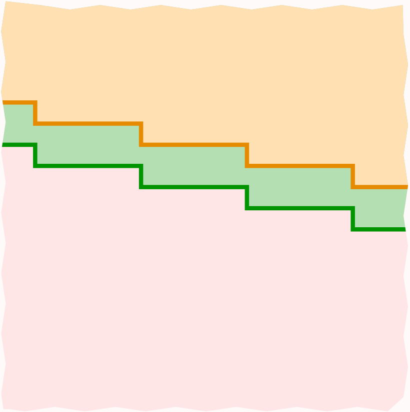

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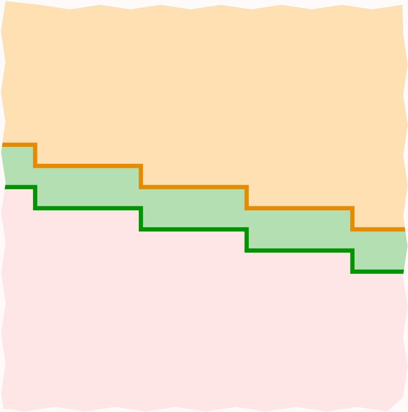

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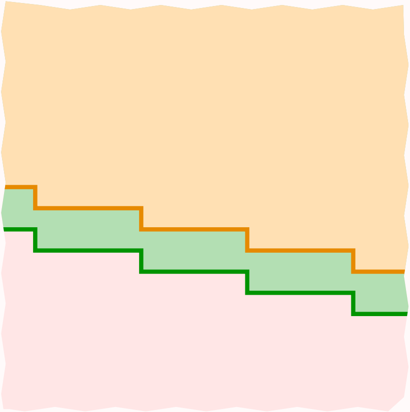

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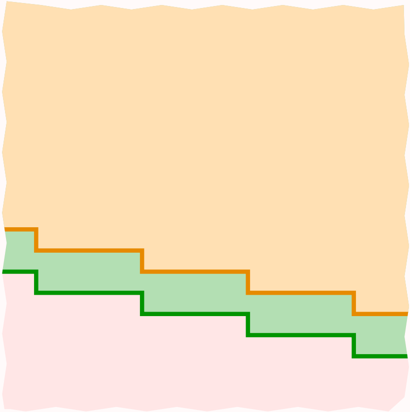

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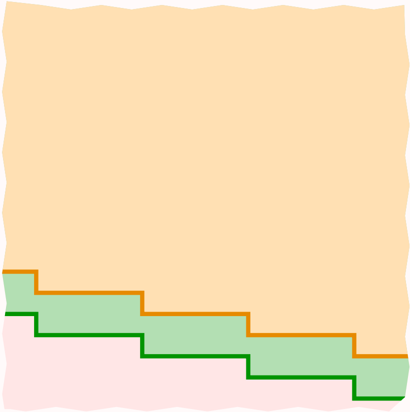

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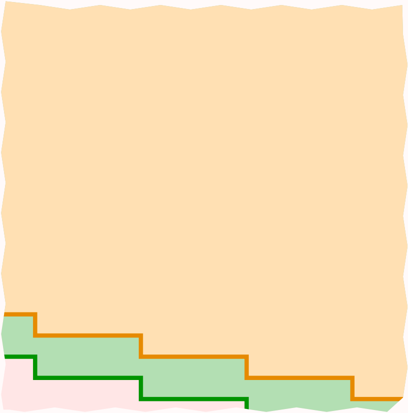

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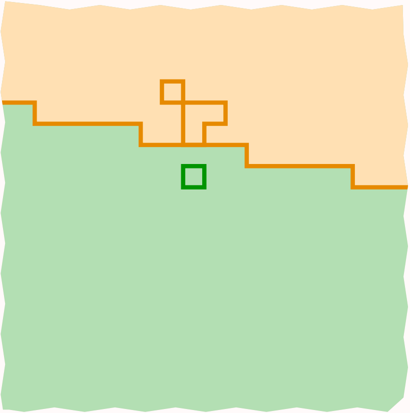

Defi: $\theta \in \mathbb{S}$ is **deterministic** if
$$\mathcal{H}_\theta := \left\{ \mathbf{v} \middle| \mathbf{v} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} < 0 \right\} \underset{\mathcal{X}}{\longrightarrow} \mathbb{Z}^2.$$

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Defi: $\theta \in \mathbb{S}$ is **deterministic** if $\mathcal{H}_\theta \underset{\mathcal{X}}{\longrightarrow} \mathbb{Z}^2$.

Prop: Equivalence:

1. θ is *deterministic*.
2. $\exists r, \mathcal{H}_\theta \cap \mathcal{B}_0(r) \underset{\mathcal{X}}{\longrightarrow} \{(0,0)\}$.
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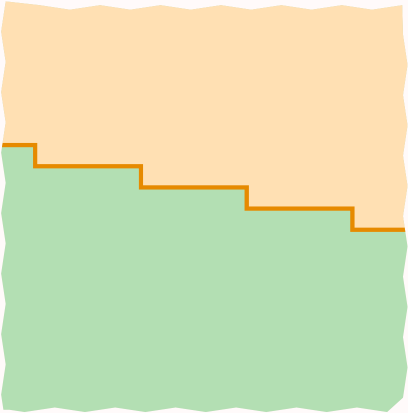

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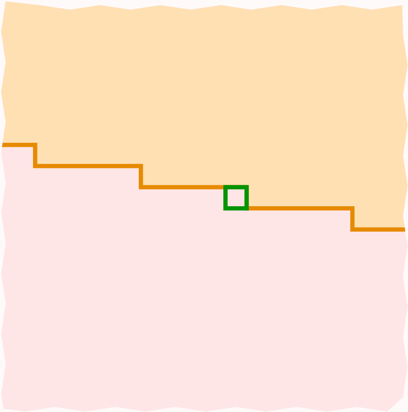

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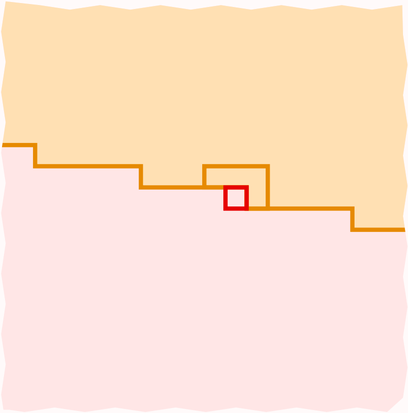

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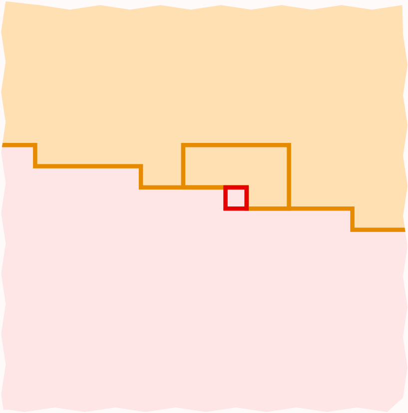

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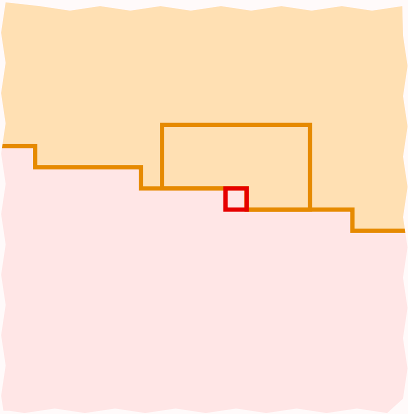

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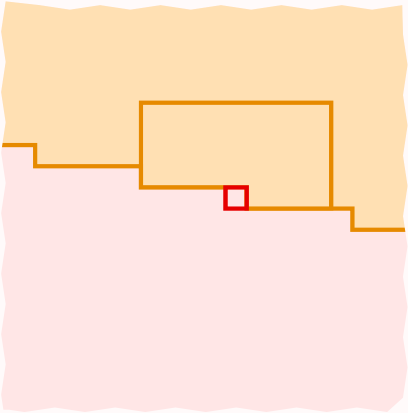

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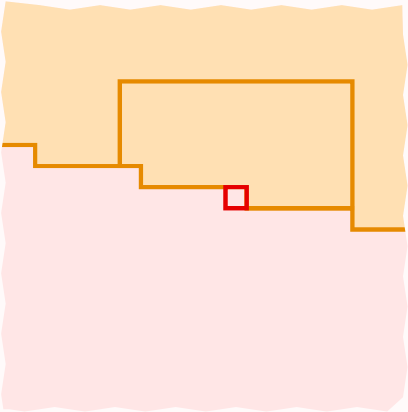

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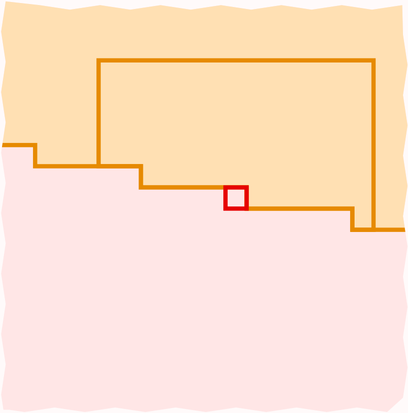

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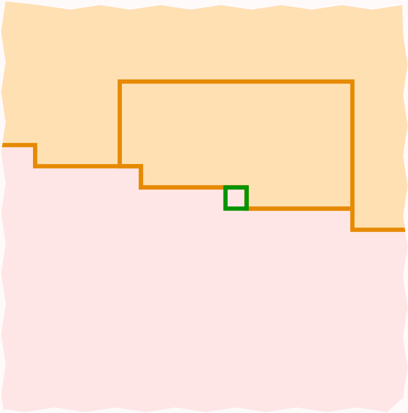
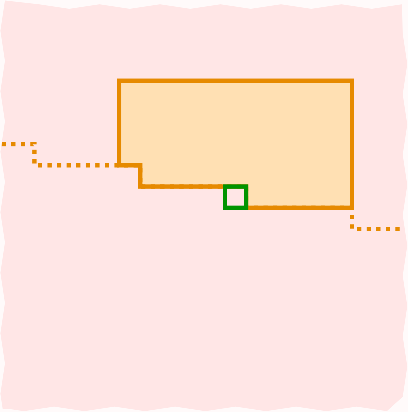


Set of deterministic directions

Defi:

- $\mathcal{D}(\mathcal{X}) \subseteq \mathbb{S}$:
  deterministic directions of $\mathcal{X}$.
- $\mathcal{D}_r(\mathcal{X}) \subseteq \mathcal{D}(\mathcal{X})$:
  directions with radius $r \in \mathbb{N}$.

Prop: $\mathcal{D}_r(\mathcal{X})$ is *open*.
$\mathcal{D}(\mathcal{X}) = \bigcup_r \mathcal{D}_r(\mathcal{X})$ is *open*.



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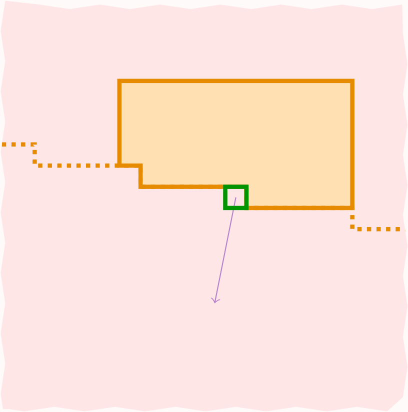

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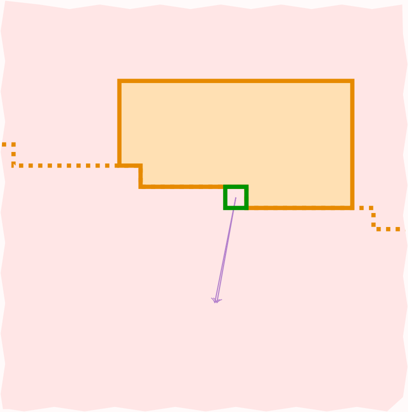

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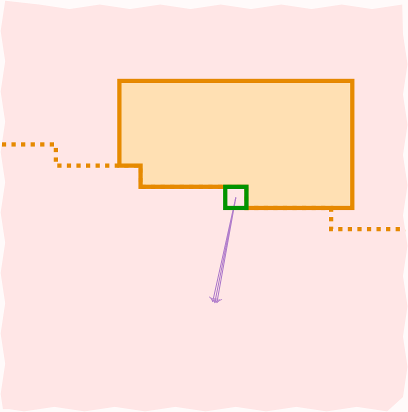

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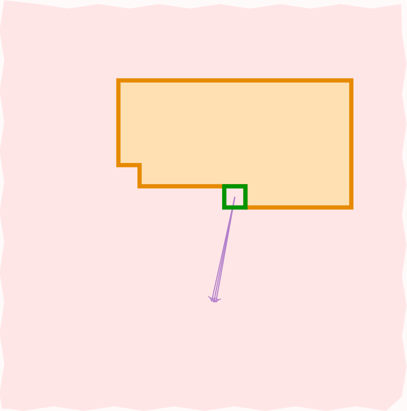
$\mathcal{X}$ effective $\Rightarrow$ $\mathcal{D}(\mathcal{X})$ *effectively open*.



Extremal permutivity

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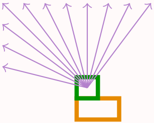

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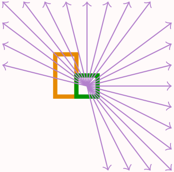

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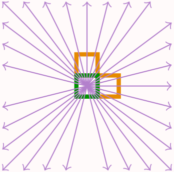

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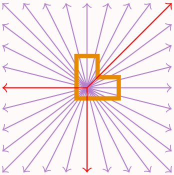

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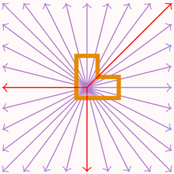
Defi: U is **extremally permutive** if
$\forall i$ extremal in $U, U \setminus \{i\} \underset{\mathcal{X}}{\longrightarrow} U$.

Rem: If U is extremally permutive,
then any $\theta \in \mathcal{D}(\mathcal{X})^C$ is outgoing orthogonal to its convex hull.



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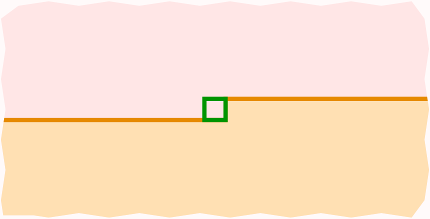

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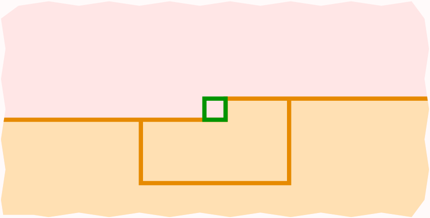

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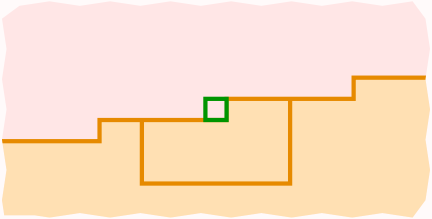

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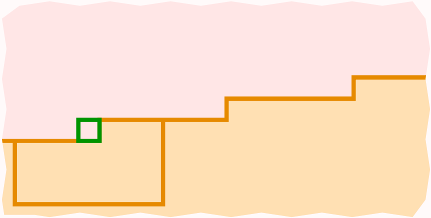

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1. $\mathcal{X}$ admits some finite *extremally permutive* shape $U \neq \emptyset$.
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In that case, $\mathcal{D}(\mathcal{X})^C \subset_{\text{finite}} \mathbb{Q}$ ($\perp$ to the boundary of U).

Determinism and periodicity

Thm: Equivalence:

1. $\mathcal{X}$ is *periodic* in direction θ.
2. $\exists U$ finite, $U \underset{\mathcal{X}}{\longrightarrow} U + \mathbb{R}_+ \mathbf{v}_\theta$.
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Thm: Equivalence:

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3. $\exists \theta, [\theta + \frac{\pi}{2}, \theta + \frac{\pi}{2}] \in \mathcal{D}(\mathcal{X})$.

[Boyle-Lind, Schwartzman]: similar result for $\mathbb{Z}^d$-actions over compact.

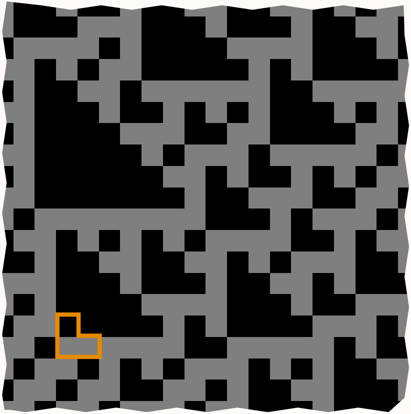
Deterministic aperiodic subshifts

Cor: Every aperiodic subshift has:

1. at least three directions of nondeterminism
   (at least two nonexpansive directions).
2. OR exactly two, opposite ones,
   other directions having unbounded radius.

[Boyle-Lind 97 / Hochman 11]:
$\forall W$ open $-W = W \Rightarrow \exists \mathcal{X}, \mathcal{D}(\mathcal{X}) \cap -\mathcal{D}(\mathcal{X}) = W$.
[G-Zinoviadis]:
if W is effectively open, then $\mathcal{X}$ can be an SFT.

Annihilator



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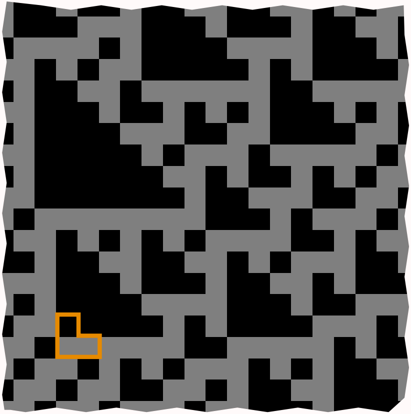
configuration $x \in \mathcal{X} \sim$ Laurent series $\sum_{i,j \in \mathbb{Z}} x_{i,j} X^i Y^j$ [Schmidt 95]

Defi: $P \in \mathbb{K}[X, Y]$ **annihilates** $\mathcal{X}$ if $\forall x \in \mathcal{X}, P.x = 0$.

e.g. $X + Y + 1 = 0$



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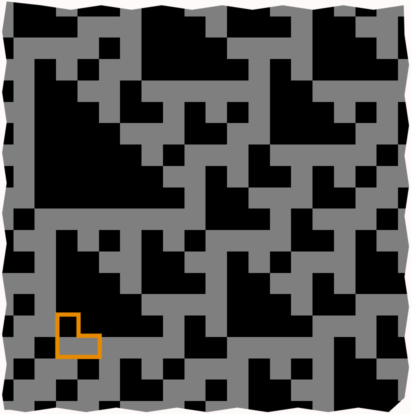

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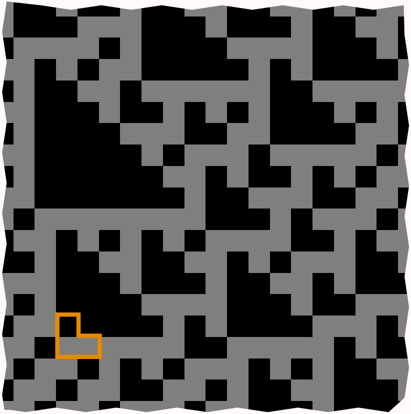

Rem:
An annihilator support is *fully permutive*.

Complexity

Defi: $U \subset_{\text{finite}} \mathcal{X}$
$\mathcal{L}_U(\mathcal{X}) := \left\{ x_{|U} \,\middle|\, x \in \mathcal{X} \right\}$: **language** of **support** $U \in \mathbb{Z}^2$ of $\mathcal{X}$.

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Defi: $U \subset_{\text{finite}} \mathbb{Z}^2$ is **Nivat** if $|\mathcal{L}_U(\mathcal{X})| \leq |U|$.

[Nivat] conjecture: $\exists$ Nivat rectangle $\Rightarrow \mathcal{X}$ is periodic.

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$\mathcal{L}_U(\mathcal{X}) := \left\{ x_{|U} \,\middle|\, x \in \mathcal{X} \right\}$: **language** of **support** $U \in \mathbb{Z}^2$ of $\mathcal{X}$.

- if $U \underset{\mathcal{X}}{\longrightarrow} V$, then $|\mathcal{L}_V(\mathcal{X})| \leq |\mathcal{L}_U(\mathcal{X})|$;
- moreover, $|\mathcal{L}_{U \cup V}(\mathcal{X})| = |\mathcal{L}_U(\mathcal{X})|$ iff $U \underset{\mathcal{X}}{\longrightarrow} V$.

Defi: $U \subset_{\text{finite}} \mathbb{Z}^2$ is **Nivat** if $|\mathcal{L}_U(\mathcal{X})| \leq |U|$.

[Nivat] conjecture: $\exists$ Nivat rectangle $\Rightarrow \mathcal{X}$ is periodic.

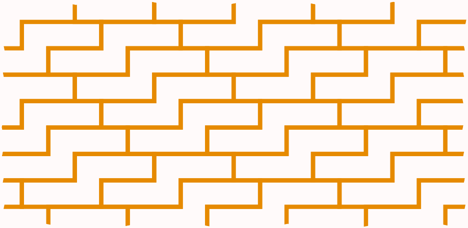
Thm:

- A minimal *Nivat* shape is *fully permutive*.
- [Kari-Szabados 16]: it contains the support of some *annihilator*.

One-tile subshift

Defi: A tile $T \subset_{\text{finite}} \mathbb{Z}^2$ **tiles** if $\exists U \subset \mathbb{Z}^2$, $T \oplus U = \mathbb{Z}^2$.

Can the only possible U be aperiodic?
[Bhattacharya 16]: No.



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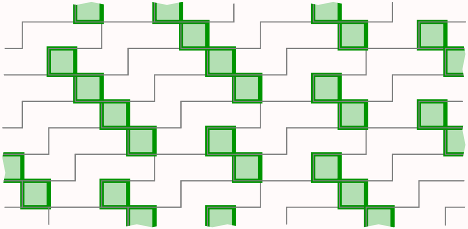

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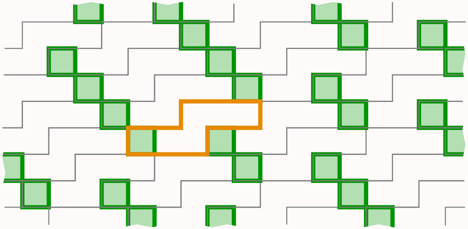

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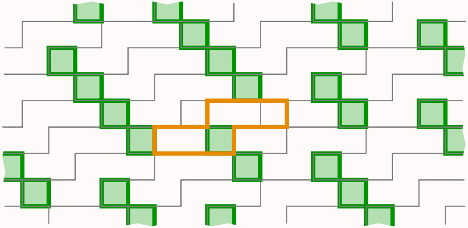

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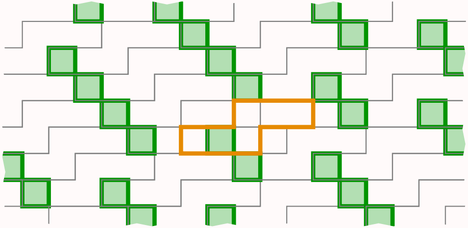

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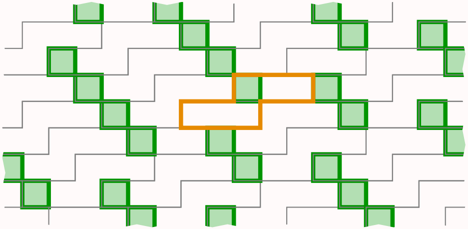

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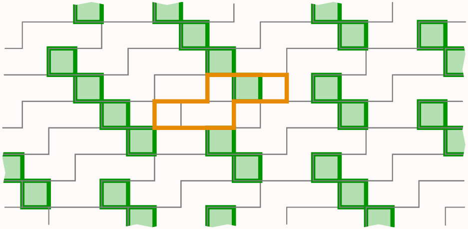

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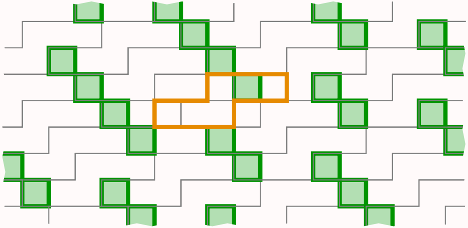

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Rem: T and $-T$ are minimal *Nivat* shapes.
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unless the convex hull admits 2 pairs of parallel sides.

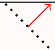
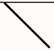
$\mathcal{X}_T \Rightarrow \exists$ Nivat shape $\Rightarrow \exists$ annihilator $\Rightarrow \exists$ fully permutive shape
$\Rightarrow \exists$ extremally permutive shape $\Leftrightarrow$ all θ are biclosing

Problems

- *Can we compute?*
  Is the **tiling problem** undecidable?
- *Can we enforce hierarchy?*
  Can we factor onto any **substitution** limit set?
- *How complex are the colons?*
  Can the **trace** be any effective subshift?
- *How does the number of distinct patterns grow?*
  Can the **entropy** be any right-computable number?
- *Which directions can be realized?*
  Can the **expansive directions** span any effectively open set?

Determinism and computation

Realizability of computationnally complex objects
in presence of deterministic directions

	universality	(directional) entropy	substitutions	traces	self-simulation
	Berger 1966	Hochman-Meyerovitch 2007	Mozes 1988	AS 2010 DRS 2010	Romashchenko-Durand-Shen 2010
	Aanderaa-Lewis 1974, Kari 1992	G-Zinoviadis 2012			Gács 2001
	Kari-Lukkarila 2008				
	Lukkarila 2009		le Gloannec-Ollinger 2012		
	✓	✓	✓	✓	✓
	$\perp$	0	periodic	periodic	trivial

Applications

Characterization of realizability for SFT:

- rectangle *substitutions*
- *traces*: all 1D **effective** subshifts
- directional *entropies*: all **right-computable** numbers
- sets of *expansive directions*: all **effectively open** sets
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Ongoing work:

- other types of growth rates
- realizing Cantor systems as subactions
- measures
- minimal SFT
- reversible (total) CA (expansiveness undecidability, ...)

Conclusion

- $\mathcal{D}(\mathcal{X})$ *structures* the 2D dynamics in terms of *1D subdynamics*
- nice properties
  (continuous directional entropy within connected components...)
- $\sim$ *Lyapunov exponents*
  $\sim$ *information flows*
- *combinatorics* (Nivat conjecture),
  algebra (annihilators),
  tilings (by disconnected polyominoes)...
- ... *CAT0-complexes*, *automata groups*, *complexity* (P vs NP)...

Conclusion

Known:

- $\mathcal{D}(\mathcal{X})$ is *effectively open* if $\mathcal{X}$ effective.
- Too much *determinism* $\Rightarrow$ essentially *1D system*.
- SFT with $\left|\mathcal{D}(\mathcal{X})^C\right| = 2$ have full computational power.
- $\mathcal{D}(\mathcal{X})$ can be any open set (effectively, if $\mathcal{X}$ SFT).

Unknown:

- Is there an *aperiodic subshift* with $\left|\mathcal{D}_r(\mathcal{X})^C\right| = 3$?
- CA with $\mathbb{Q} \subset \mathcal{D}(\mathcal{X})$?
- Nivat conjecture?
- Aperiodic 3D one-tile system?
- sofic w/ $\mathcal{D}(\mathcal{X}) \neq \emptyset \Rightarrow$ effective w/ $\mathcal{D}(X) \neq \emptyset \Rightarrow$ effective w/ entr.dim. ≤ 1