Distortion

in cellular automata and one-head machines

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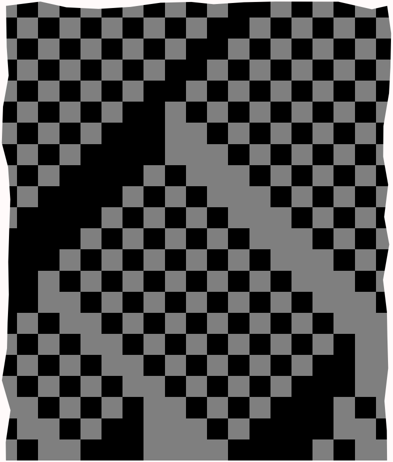
Cellular automata

alphabet : finite set A
configuration : $x \in A^{\mathbb{Z}}$
shift: $\sigma(x)_i = x_{i+1}$

cellular automaton:

$$F : A^{\mathbb{Z}} \to A^{\mathbb{Z}} \begin{cases} \text{continuous} \\ \sigma F = F\sigma \end{cases}$$



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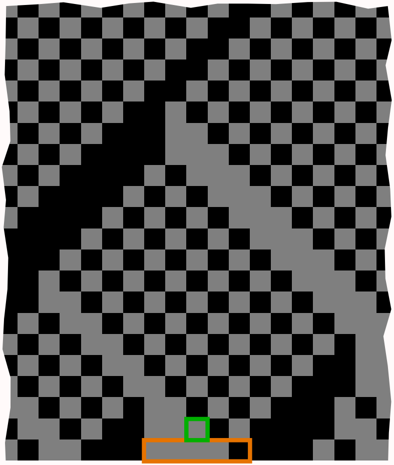

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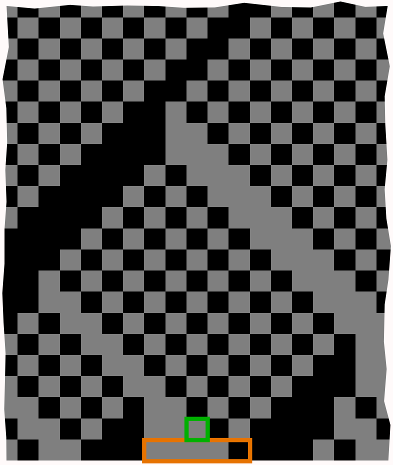
radius of F: minimal r



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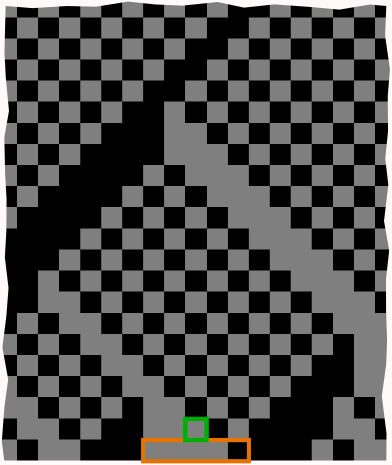

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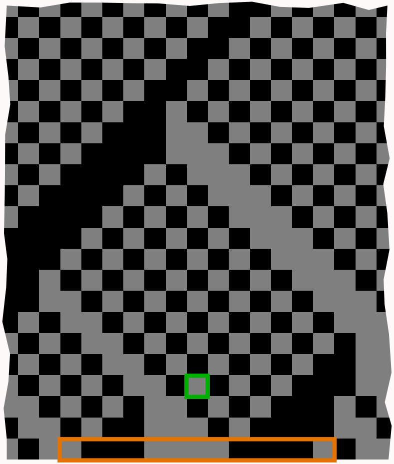

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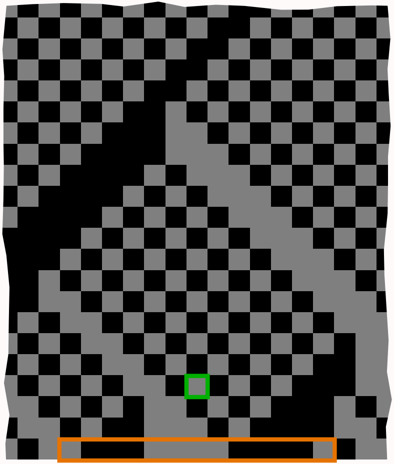

$\forall t$, F^t is a CA, with radius $\leq tr$

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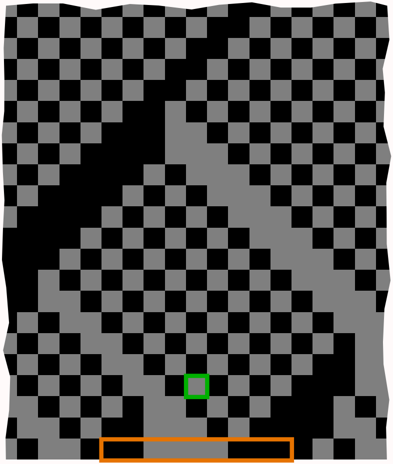

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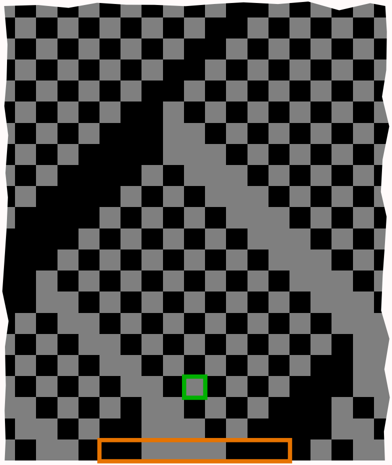

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subshift: $X \subset A^{\mathbb{Z}}$ closed, $\sigma(X) = X$
cellular automaton over $X \subset A^{\mathbb{Z}}$:

$$F : X \to X \begin{cases} \text{continuous} \\ \sigma F = F\sigma \end{cases}$$

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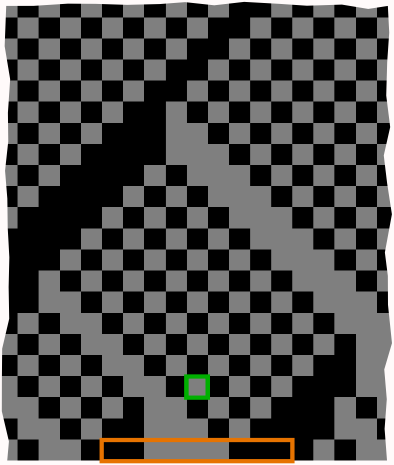

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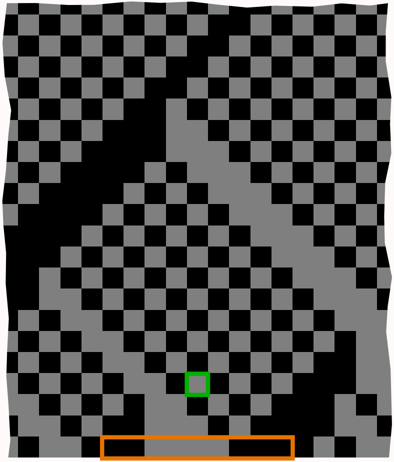

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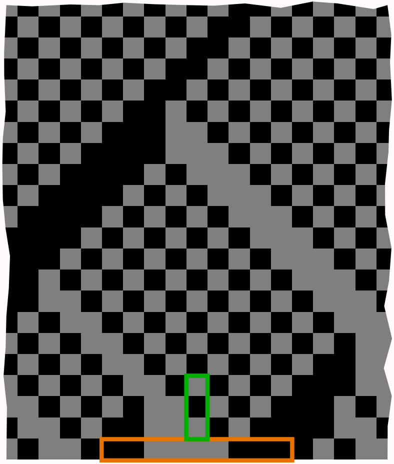

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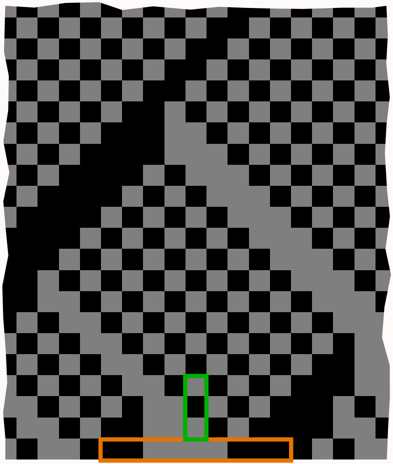

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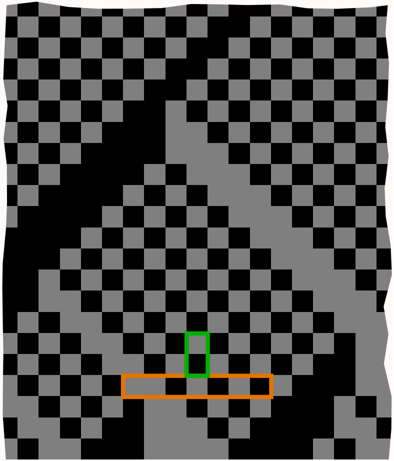

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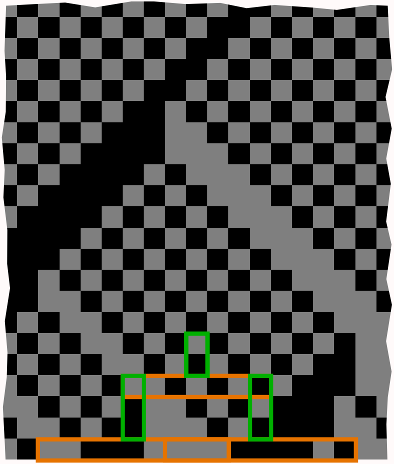

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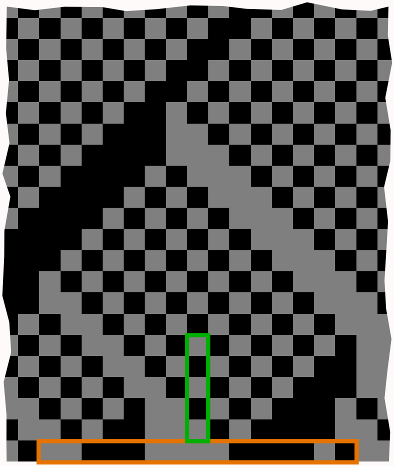

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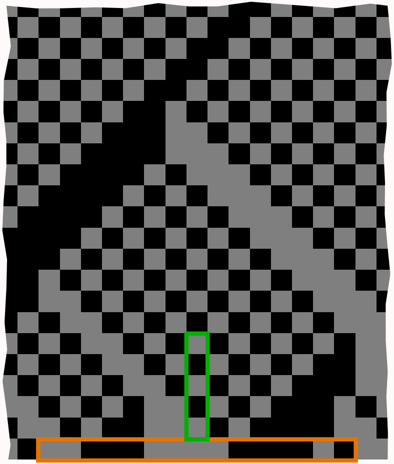

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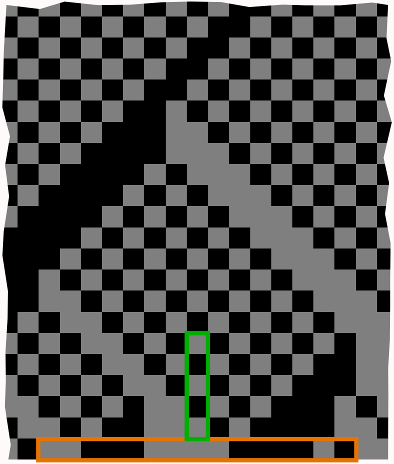

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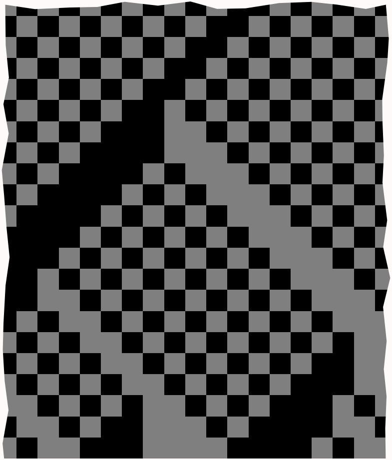

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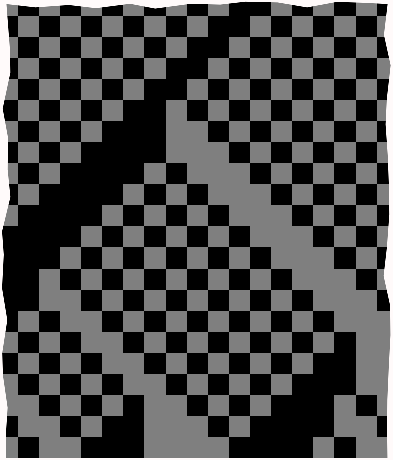

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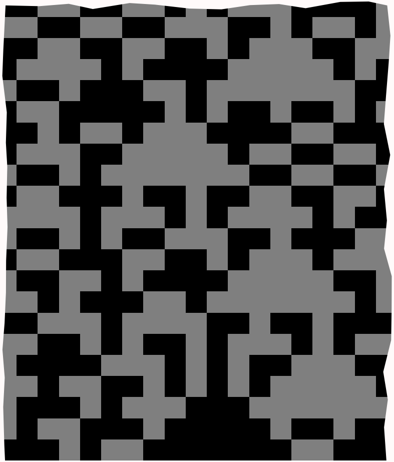

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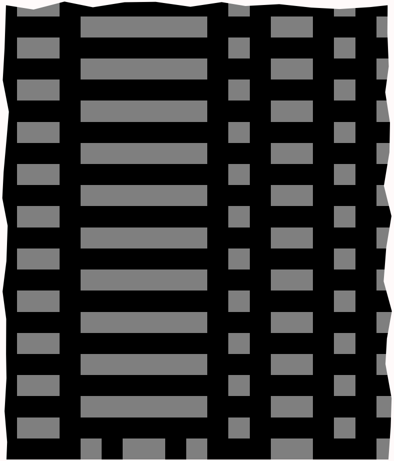

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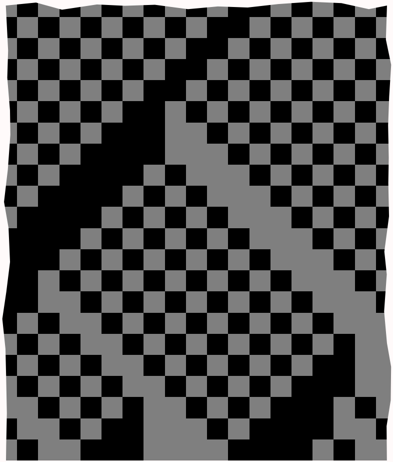

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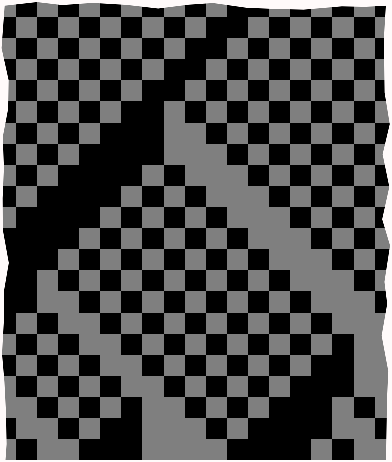

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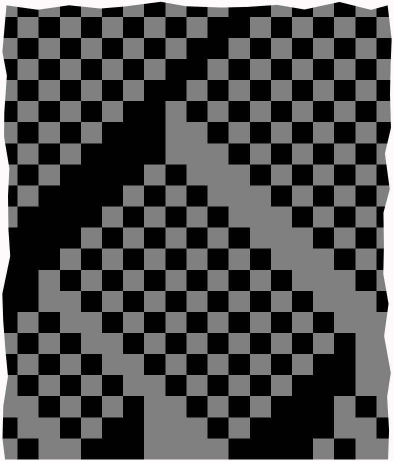

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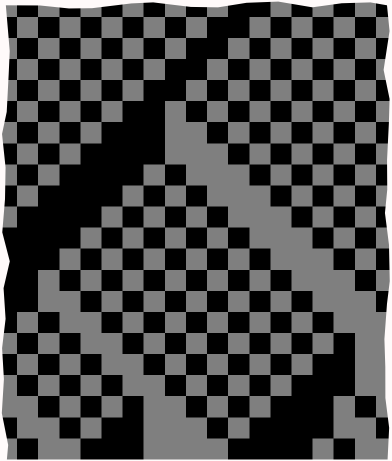
Lower gap



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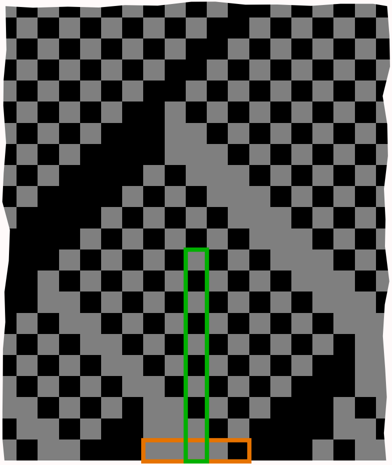

trace: $T_F(x) = (F^t(x)_0)_{t\in\mathbb{N}}$

$T_F(x)_{0\dots t}$ depends only on $x_{-r_t\dots r_t}$,

where $D_t = 2r_t + 1$.

so $\left|A^{D_t}\right| \geq |\{T_F(x)_{0\dots t} | x \in X\}|$

$> t$, unless $T_F(X)$ is preperiodic

[Morse-Hedlund]

[Cyr-Franks-Kra-Petite 2016]

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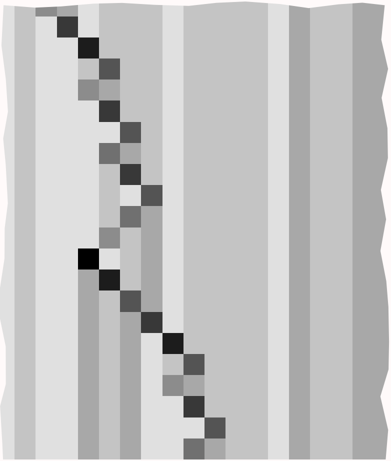
One-head machines

one-head machine: $\mathcal{M} = (Q, \Sigma, \delta)$:

- Q: set of **states**
- Σ: alphabet of **symbols**
- $\delta \subset \begin{array}{c} (Q \times \{\pm 1\} \times Q) \\ \sqcup \; (Q \times \Sigma \times Q \times \Sigma) \end{array}$

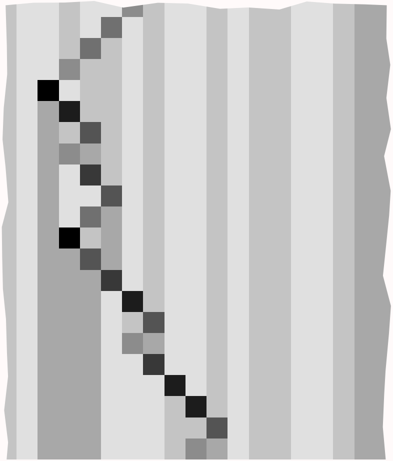


$\longrightarrow$ "CA" $F_{\mathcal{M}}$ over

$X_{\mathcal{M}} = \left\{ x \in (\Sigma \sqcup (Q \times \Sigma))^{\mathbb{Z}} \middle| \, |\{ i | x_i \notin \Sigma\}| \leq 1 \right\}$

D_t= max number of *visited cells* after t steps [Jeandel 2013]

Upper gap



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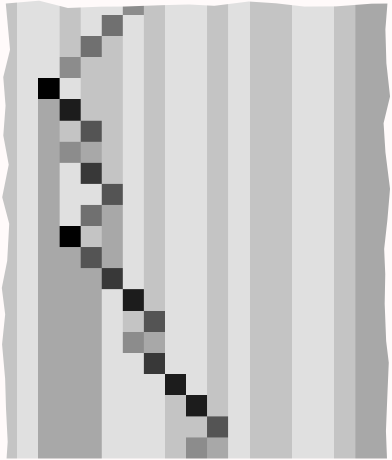

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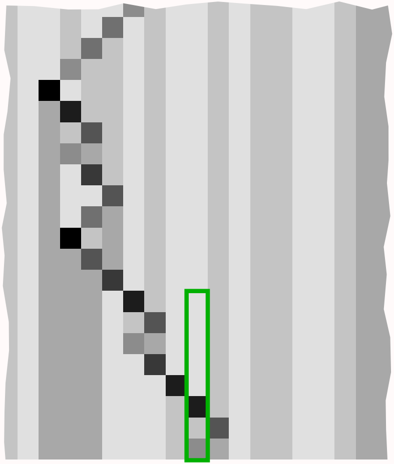
crossing sequence $u_{i,t}$:
head *states* visiting i before step t.
[Hennie 1965, Jeandel 2013]
$|u_{i,t}|$: number of *visits* in i



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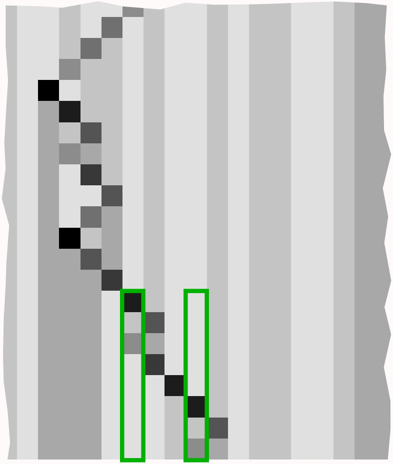

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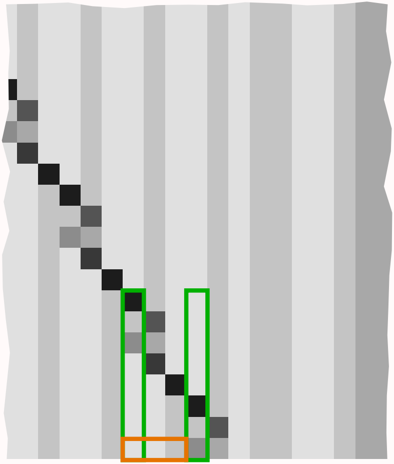

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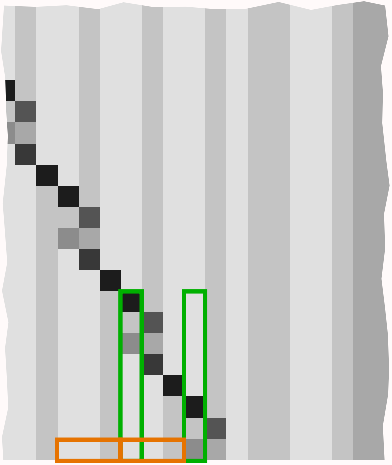

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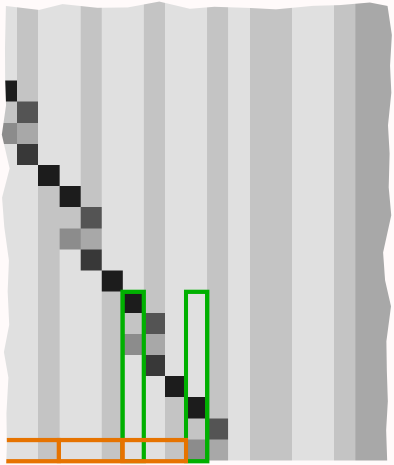

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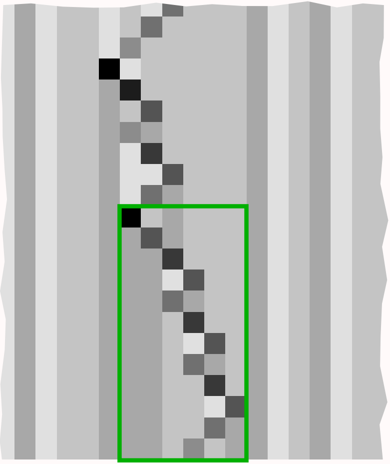
  repeat $x_{i\ldots j-1}$ *periodically*
  $\longrightarrow$ speed > 0
- else all $u_{i,t}$ cannot be short
  $t = \sum_{i<D_t} |u_{i,t}| = \Theta(D_t \log D_t)$
  $\longrightarrow D_t = O(t/\log t)$



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Distorted machine over a subshift

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$\sim$[Hochman-Lind 2011]

Aperiodic machines

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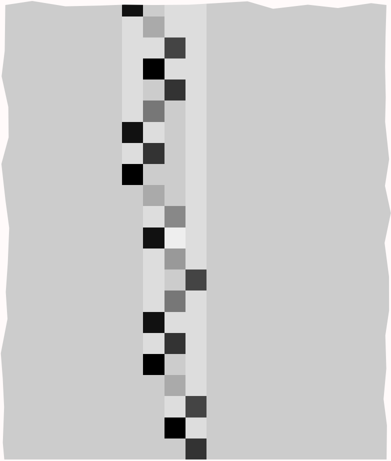

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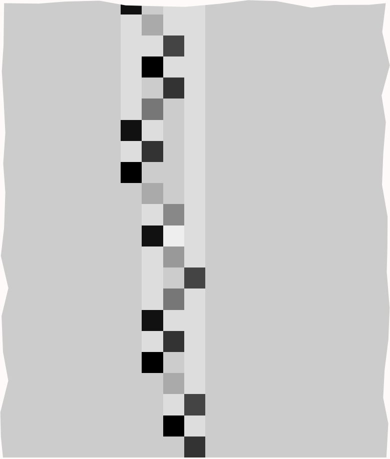

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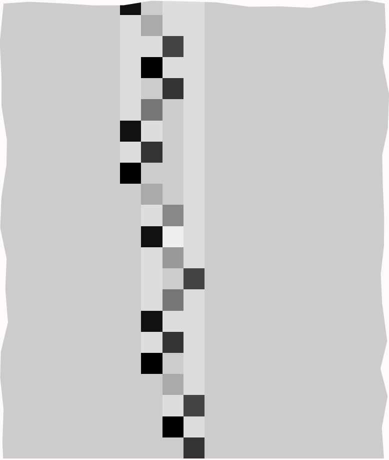

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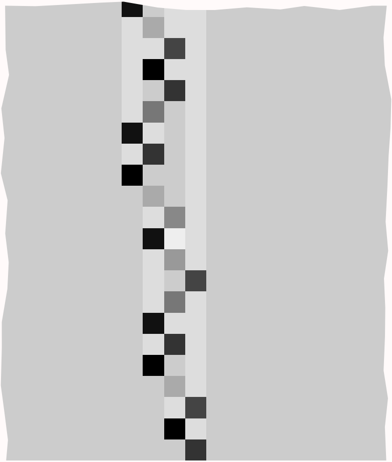

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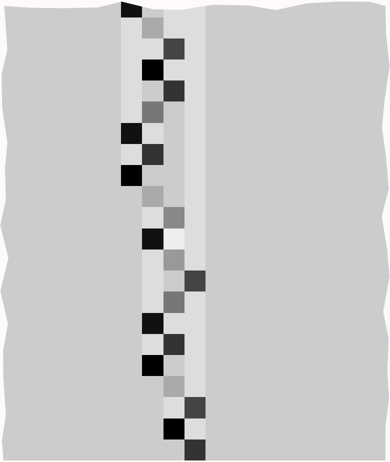

Embedding TM into CA

∃ distorted CA over the *full* shift

$\longleftarrow$ extend $F_{\mathcal{M}}$ into F, i.e. $F_{|X_{\mathcal{M}}} = F_{\mathcal{M}}$, with $D_t(F) = D_t(F_{|X_{\mathcal{M}}})$:

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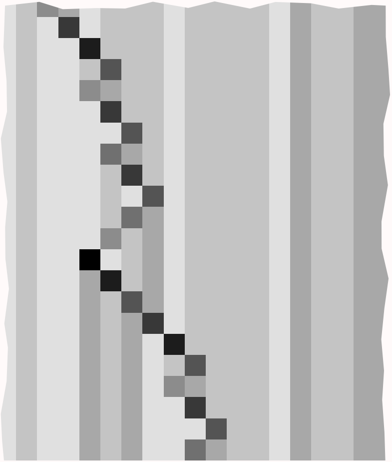

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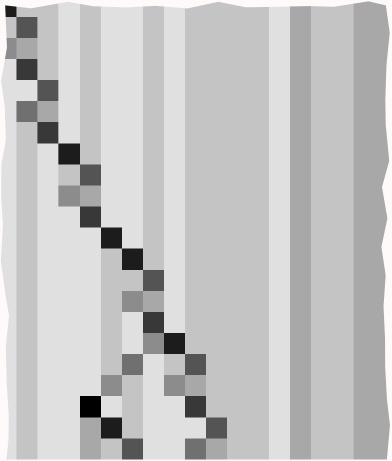

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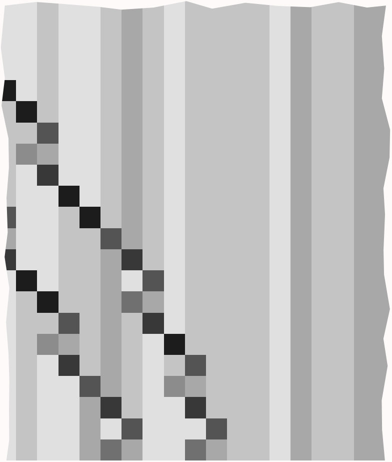

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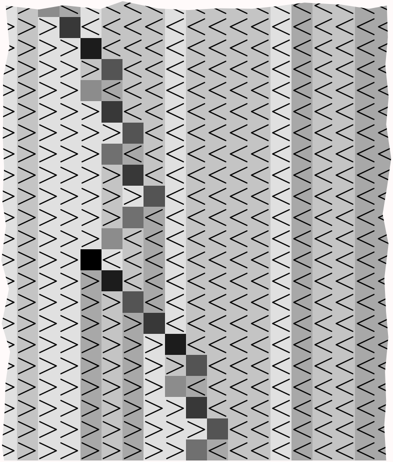

SFT: set of configurations avoiding finitely many patterns

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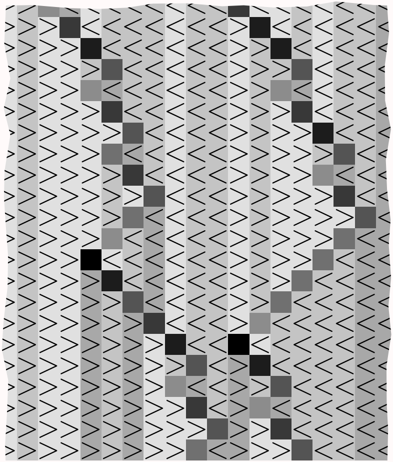

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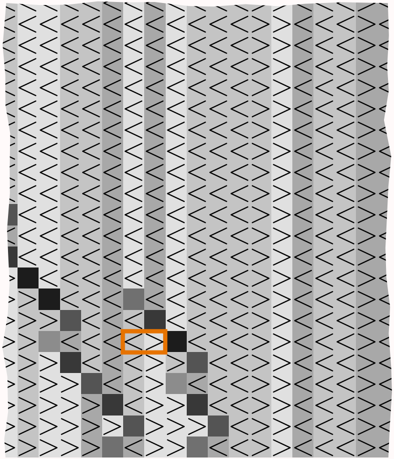

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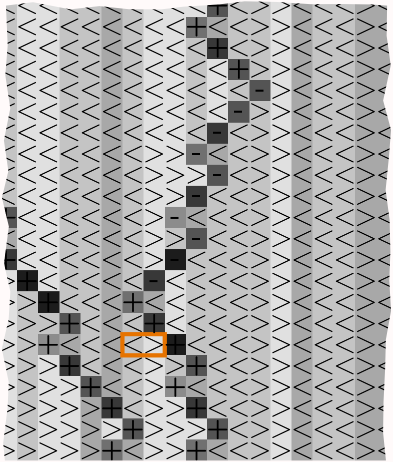

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Embedding TM into CA

∃ distorted CA over the *full* shift
/ over any *uncountable sofic* subshift

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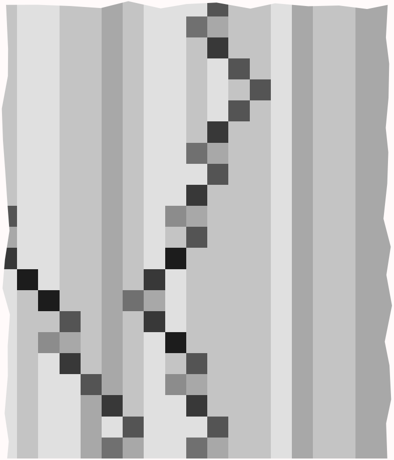

SFT: set of configurations avoiding finitely many patterns
sofic subshift: set of path labels in a finite graph
⟶ countable sofic subshift: all s.c.c. are cycle

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$\longrightarrow$ *Distortion* is *undecidable*,
given a CA over any fixed uncountable sofic subshift.

Neighborhoods

cellular automaton:

$$F : A^{\mathbb{Z}} \to A^{\mathbb{Z}} \begin{cases} \text{continuous} \\ \sigma F = F\sigma \end{cases}$$

$\longleftrightarrow$ $\exists N$ finite neighborhood

neighborhood: interval $N \subset \mathbb{Z}$ s.t. $x_{|N} = y_{|N} \Rightarrow F(x) = F(y)$.
I: intersection of possible N

I may *not* be a neighborhood!
How does I evolve for F^t?

$|I|$ can grow close to log, even when D_t grows close to linear.

Periodized mud runner

neighborhood: interval $N \subset \mathbb{Z}$ s.t.
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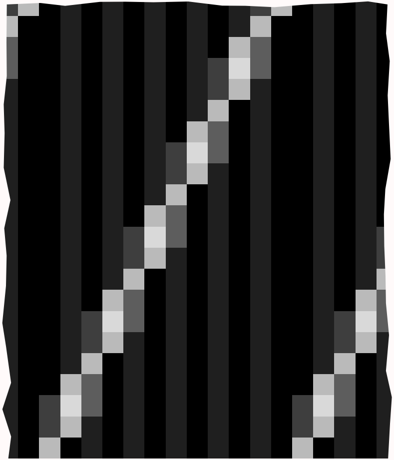

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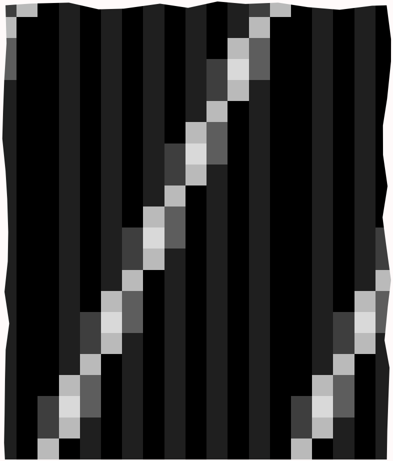
$\exists F$, $|I|$ grows close to $\log t$,
but D_t grows close to t.



Conclusion

Families of distorted cellular automata over subshifts:

- *One-head machines* ($\rightarrow$ full shifts) [Cassaigne, Kari-Ollinger]
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Perspectives:

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  Study eigenspaces from the substitution behind SMART?
  [Cassaigne-Ollinger-Torres 2017]
- *group distortion*?
  Does there exist F such that F^t can be expressed as a length-$o(t)$ composition of CA from a fixed finite family?
  [Cyr-Franks-Kra-Petite 2016]