Realizability of non-expansive dynamics and applications

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General setting

Consider an action by homeomorphisms

$$T : G \curvearrowright X \subset \{0,1\}^A$$

where:

- G is a countable group
- A is a countable set (usually $\mathbb{N}, \mathbb{Z}$ or G).
- X is closed for the product topology.

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$\triangleright$ The goal of this talk is to study under which conditions these actions can be recovered as subactions of simpler dynamical systems (SFTs and sofic subshifts).

General setting

Odometer $T : \mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{N}}$ "addition in base 2 with right carry"

If $x = 1111\ldots$ then $T(x) = 0000\ldots$. Otherwise let $k(x)$ be the index of the first 0 in x. Then:

$$T(x)_n = \begin{cases} 1 \text{ if } n = k(x) \\ 0 \text{ if } n < k(x) \\ x_n \text{ if } n > k(x) \end{cases}$$

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$$\begin{aligned} x &= 010010100010001000\ldots \\ T(x) &= 110010100010001000\ldots \\ T^2(x) &= 001010100010001000\ldots \\ T^3(x) &= 101010100010001000\ldots \\ T^4(x) &= 011010100010001000\ldots \\ T^5(x) &= 111010100010001000\ldots \\ T^6(x) &= 000110100010001000\ldots \end{aligned}$$

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Full G-shift

Let $\sigma : G \curvearrowright \{0,1\}^G$ be given by:

$$\sigma^h(x)_g = x_{h^{-1}g}.$$

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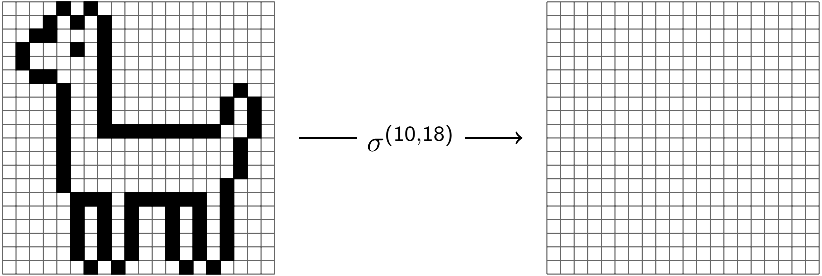

Figure: A random configuration $x \in \{■, □\}^{\mathbb{Z}^2/20\mathbb{Z}^2}$ and its image by $\sigma^{(10,18)}$.

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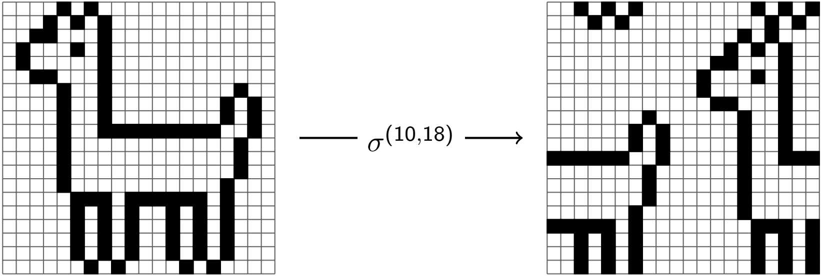

Figure: A random configuration $x \in \{\blacksquare, \square\}^{\mathbb{Z}^2/20\mathbb{Z}^2}$ and its image by $\sigma^{(10,18)}$.

General setting

$\phi : \mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}^d}$ (Invertible) cellular automaton

Let $F \subset \mathbb{Z}^d$ be a finite set and $\Phi : \{0,1\}^F \to \{0,1\}$ a function. Let

$$\phi(x)_v = \Phi(\sigma^{-v}(x)|_F) \text{ “=” } \Phi(x|_{v+F}).$$

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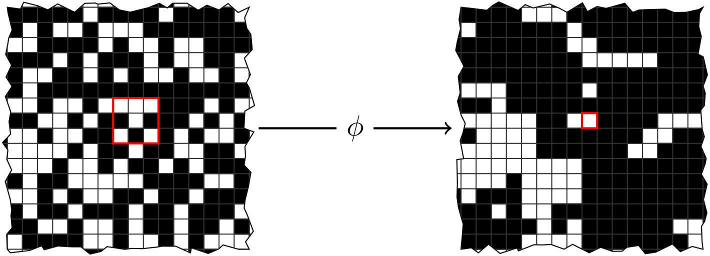

*This example is not invertible.

Definitions

Let $\mathcal{A}$ be a finite alphabet.

Definition: full G-shift

The full G-shift is the action $\sigma : G \curvearrowright \mathcal{A}^G$ where:

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Examples:

- $X = \left\{x \in \{0,1\}^{\mathbb{Z}} \mid \text{no two consecutive 1's in } x\right\}$
- $X = \left\{x \in \{0,1\}^{\mathbb{Z}^2} \mid \text{finite CC of 1's are of even length}\right\}$

Definitions

Luckily, subshifts can also be described in a combinatorial way.

- A *pattern* is a finite configuration, i.e. $p \in \mathcal{A}^F$ where $F \subset G$ and $|F| < \infty$. We denote $\mathrm{supp}(p) = F$.
- A *cylinder* is the set $[a]_g := \{x \in \mathcal{A}^G \mid x_g = a\}$.
- $$[p] := \bigcap_{g \in \mathrm{supp}(p)} [p_g]_g.$$

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Proposition

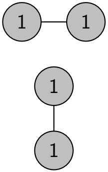
A subshift is a set of configurations avoiding patterns from a set $\mathcal{F}$.

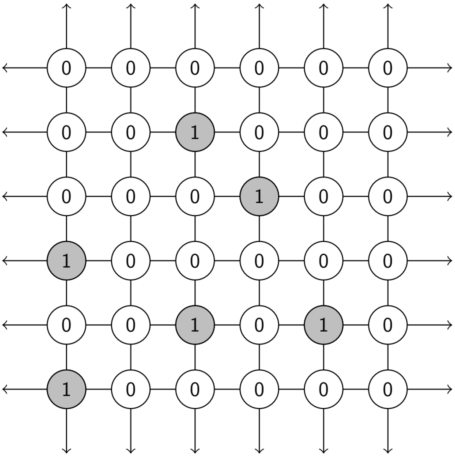
$$X = X_{\mathcal{F}} := \mathcal{A}^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} \sigma^g([p])$$

Example in $\mathbb{Z}^2$: Hard-square shift

Example: Hard-square shift. X is the set of assignments of $\mathbb{Z}^2$ to $\{0, 1\}$ such that there are no two adjacent ones.

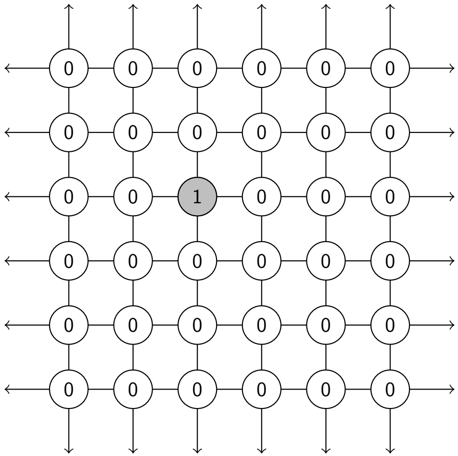
$\mathcal{F} =$



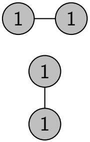


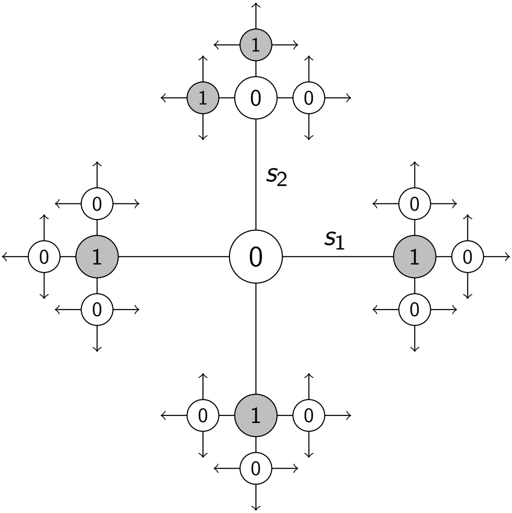
Example: one-or-less subshift.

$$X_{\leq 1} := \{x \in \{0,1\}^G \mid 0 \notin \{x_u, x_v\} \implies u = v\}.$$



Example: Same rule as hard-square in F_2.

$\mathcal{F} =$ 



Simple classes of subshifts

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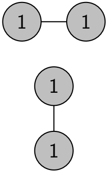
Definition: sofic subshift

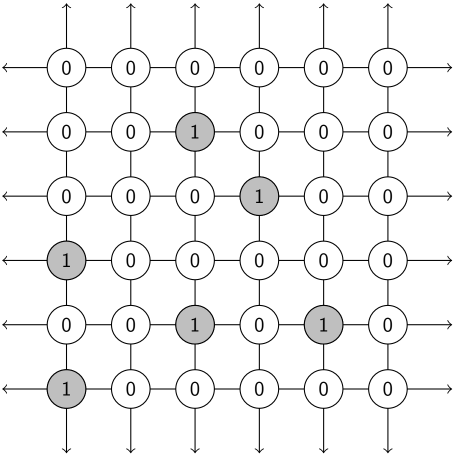
A *sofic subshift* is the image of an SFT via a shift-commuting continuous map.

Definition: effective subshift

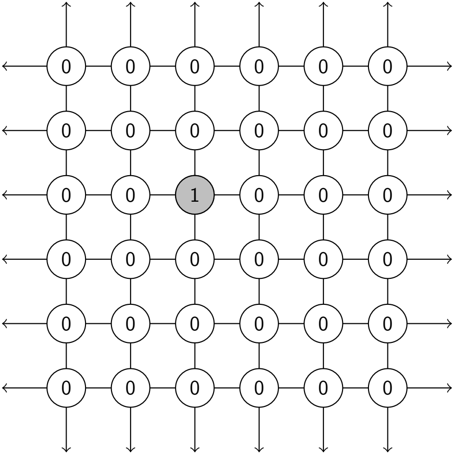
An *effectively closed subshift* is a subshift that can be defined by a recursively enumerable set of forbidden patterns.

Example SFT: Hard-square shift

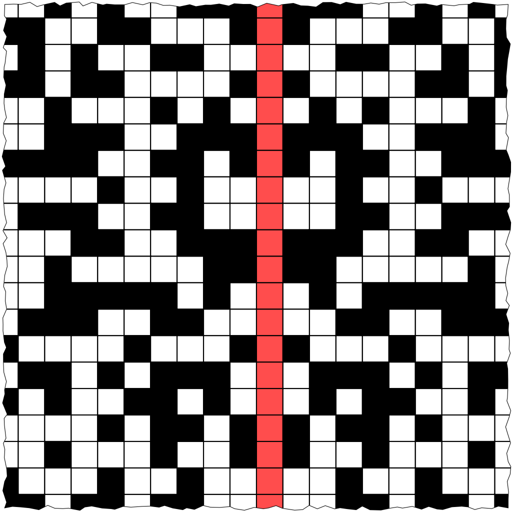
$\mathcal{F} =$ 



Example sofic: one-or-less subshift (in $\mathbb{Z}^2$)



Effectively closed subshift: Mirror shift



What about the subactions of these classes?

Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $H \leq \mathbb{Z}^d$.

- ▷ What can we say about the system $(X, \sigma|_H)$?
- ▷ Same question when X is an SFT, sofic or effectively closed.

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Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and $H \leq \mathbb{Z}^d$.

▷ What can we say about the system $(X, \sigma|_H)$?
▷ Same question when X is an SFT, sofic or effectively closed.

Remark: Subshifts are expansive, subactions not necessarily

▷ Let $(\mathbb{Z}, 0) \leq \mathbb{Z}^2$ and the sequence

$$(y_n)_v = \begin{cases} 1 \text{ if } v = (0, n) \\ 0 \text{ else} \end{cases}$$

But: $\sup_{z \in \mathbb{Z}} d(\sigma^{(z,0)}(y_n), \sigma^{(z,0)}(y_m)) \leq 2^{-\min(n,m)}$

Question 1: What type of systems can we obtain as subactions?

Effectively closed Cantor set

$X \subset \{0,1\}^A$ is *effectively closed* if $X = \{0,1\}^A \setminus \bigcup_{w \in L}[w]$ where L is a recursively enumerable language.

Effectively closed dynamical system

$X \subset (\{0,1\}^A)^G$ is an *effectively closed dynamical system* if it is an effectively closed Cantor set and G acts by shifts.

Question 1: What type of systems can we obtain as subactions?

This gives a nice way of interpreting actions:

Effectively closed action

For $T : G \curvearrowright X \subset \{0,1\}^A$ consider $Y \subset \{0,1\}^{A \times G}$ defined by:

$$Y = \left\{ y \in \{0,1\}^{A \times G} \text{ such that } \begin{array}{l} y|_{A \times \{1_G\}} \in X \\ y|_{A \times \{g\}} = T^g(y|_{A \times \{1_G\}}) \end{array} \right\}.$$

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Theorem (Hochman)

Every subaction of an effectively closed subshift (also sofic/SFT) is an effectively closed dynamical system.
Proof: blackboard.

Question 2: can we realize any action as a subaction of a subshift?

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However, we will see later that the 2-odometer can be obtained as a factor of a subaction of an SFT!

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Answer: Yes.

Theorem (Hochman)

For every effectively closed action $T : \mathbb{Z}^d \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ there exists a $\mathbb{Z}^{d+2}$-SFT $\hat{X}$ such that one of its $\mathbb{Z}^d$-subactions is an extension of T.

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$$\begin{array}{lccc}
\mathbb{Z}^{d+2} & (\hat{X}, \sigma) & & \\
 & \text{subaction} \Big\downarrow & & \\
\mathbb{Z}^{d} & (\hat{X}, \sigma|_{\mathbb{Z}^d}) & \xrightarrow[\text{factor}]{\qquad\qquad} & (X, T)
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Moreover, the factor is small, it is an ATIE (almost trivial isometric extension)

Question 3: can we go the other way around?

$$
\begin{array}{cccc}
\mathbb{Z}^{d+2} & (\hat{X}, \sigma) & \xrightarrow[\text{symb factor}]{} & (\hat{Y}, \sigma) \\
& \text{subaction} \downarrow & & \text{subaction} \downarrow \\
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Answer: Yes. For the expansive case

Theorem (Hochman)

If T is an effectively closed expansive $\mathbb{Z}^d$-action (i.e. conjugate to a subshift) then it is the subaction of a $\mathbb{Z}^{d+2}$-sofic subshift.
Proof idea: blackboard.

Question 4: in the expansive case, can we get rid of the factor?

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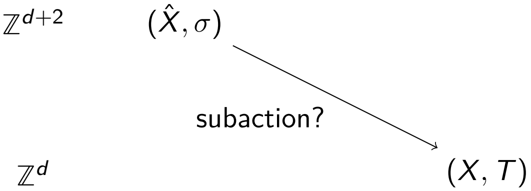

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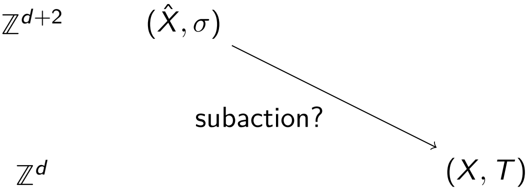

Answer: No.

Example (Hochman)

The fixed point of the Chacon substitution

$$0 \to 0010, \qquad 1 \to 0$$

generates an effectively closed subshift which is not the subaction of any $\mathbb{Z}^d$-SFT.
(Actually, any minimal e.c. $\mathbb{Z}$-subshift with $\mathrm{Aut}(X) \cong \mathbb{Z}$ works)

Question 5: can we reduce the dimension?

$$\begin{array}{lcl}
\mathbb{Z}^{d+2} & (\hat{X}, \sigma) & \\
& \text{subaction} \Big\downarrow & \\
\mathbb{Z}^{d} & (\hat{X}, \sigma|_{\mathbb{Z}^d}) & \xrightarrow[\text{factor}]{} (X, T)
\end{array}$$

Question 5: can we reduce the dimension?

$\mathbb{Z}^{d+1}$ $(\hat{X}, \sigma)$

subaction

$\mathbb{Z}^{d}$ $(\hat{X}, \sigma|_{\mathbb{Z}^d})$ $\twoheadrightarrow$ (X, T)

factor

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Example (Jeandel)

The mirror shift seen as a $\mathbb{Z}$-action over a Cantor set is not the factor of a subaction of any $\mathbb{Z}^2$-SFT.

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subaction $\downarrow$

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Answer: No.

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The mirror shift seen as a $\mathbb{Z}$-action over a Cantor set is not the factor of a subaction of any $\mathbb{Z}^2$-SFT.

However, if we restrict to the expansive case...

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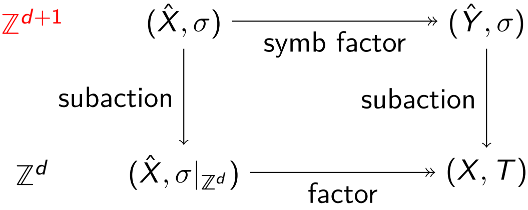

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Answer: Yes!

Theorem (Aubrun-Sablik, Durand-Romaschenko-Shen)

Every effectively closed $\mathbb{Z}^d$-subshift is the subaction (projective subaction) of a $\mathbb{Z}^{d+1}$-sofic subshift.

Panorama

	$\mathbb{Z}^{d+1}$	$(\hat{X}, \sigma)$	$\xrightarrow{\text{symb factor}}$	$(\hat{Y}, \sigma)$
expansive				subaction $\downarrow$
	$\mathbb{Z}^{d}$			(X, T)
	$\mathbb{Z}^{d+2}$	$(\hat{X}, \sigma)$		
non-expansive		subaction $\downarrow$		
	$\mathbb{Z}^{d}$	$(\hat{X}, \sigma\vert_{\mathbb{Z}^d})$	$\xrightarrow{\text{factor}}$	(X, T)

some questions

- In the non-expansive case: The factor is an ATIE.

$$(\hat{X}, \sigma|_{\mathbb{Z}^d}) \twoheadrightarrow (X, T) \times (W, S) \twoheadrightarrow (X, T).$$

The first factor is a.e. 1-1 for every invariant measure, the second is the projection, and (W, S) is an isometric action (i.e. odometer).

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▷ Can we do better with the factor?

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  $\triangleright$ Can we do better with the factor?
- In the expansive case: Which are the systems that arise as subactions of SFTs?
  $\triangleright$ Partial answers by Pavlov and Schraudner and by Sablik and Schraudner.

Why is this thing useful?

The philosophy behind it

Finitely presented group

A group G is finitely presented if it can be described as $G = \langle S|R\rangle$ where both S and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1}\rangle$$

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Recursively presented group

A group G is recursively presented if it can be described as $G = \langle S | R \rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursive sets.

$$L = \langle a, t \mid (at^n at^{-n})^2, n \in \mathbb{N} \rangle$$

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Finitely presented group

A group G is finitely presented if it can be described as $G = \langle S | R \rangle$ where both S and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$$

Recursively presented group

A group G is recursively presented if it can be described as $G = \langle S | R \rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursive sets.

$$L = \langle a, t \mid (at^n at^{-n})^2, n \in \mathbb{N} \rangle$$

$$\bigoplus_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z} \cong \langle a_n, n \in \mathbb{N} \mid \{a_n^2\}_{n \in \mathbb{N}}, [a_j, a_k]_{j,k \in \mathbb{N}} \rangle.$$

The philosophy behind it

Theorem (Highman 1961)

For every recursively presented group H there exists a finitely presented group G such that H is isomorphic to a subgroup of G.

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Corollary [Theorem: Novikov 1955, Boone 1958]

There are finitely presented groups with undecidable word problem

Just apply Highman's theorem to
$G = \langle a, b, c, d \mid b^{-n}ab^n = c^{-n}dc^n, n \in \mathtt{HALT} \rangle$... done!

An application: strongly aperiodic subshifts

Definition (Strongly aperiodic subshift)

A subshift $X \subset \mathcal{A}^G$ is *strongly aperiodic* if the shift action is free

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

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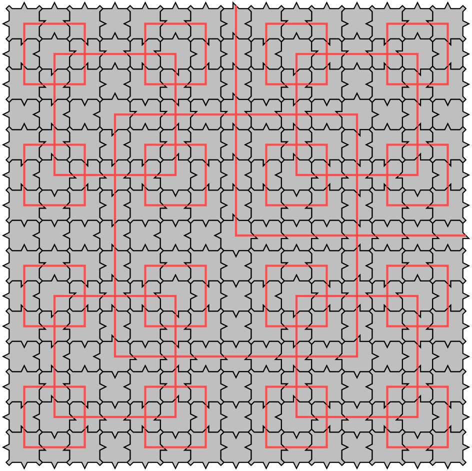
Proposition

Every 1D non-empty SFT contains a periodic configuration.

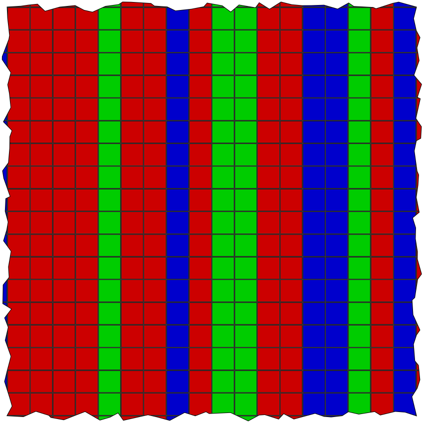
Theorem (Berger 1966, Robinson 1971, Kari 1996, Jeandel & Rao 2015)

There exist strongly aperiodic SFTs on $\mathbb{Z}^2$.

Example of strongly aperiodic $\mathbb{Z}^2$-SFT: Robinson tileset



The case of subshifts



$\in Y$

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So... why is simulation important?

It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

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Example

Let x be a fixed point of the Thue-Morse substitution.

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

Then $X = \overline{\mathrm{Orb}_\sigma(x)}$ is strongly aperiodic and effectively closed.

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Example

A Sturmian subshift given by a computable slope α.

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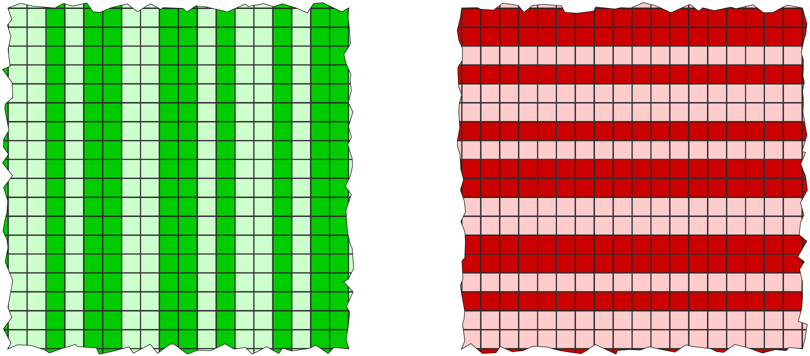

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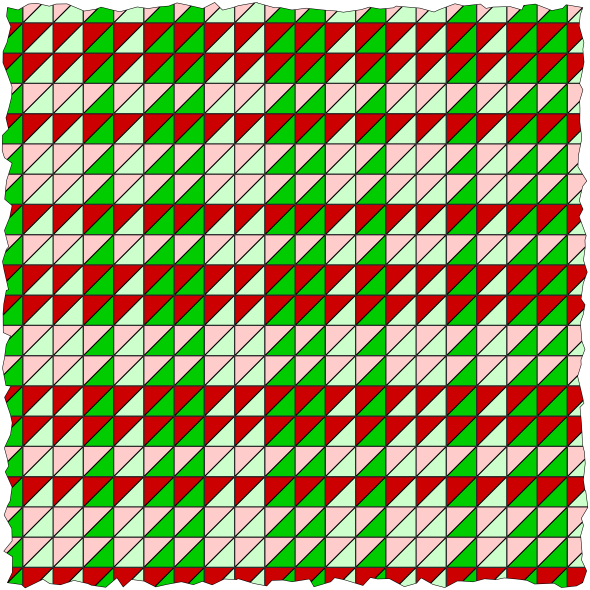

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- Easy construction of strongly aperiodic $\mathbb{Z}^2$-SFTs
- $\mathbb{Z}^2$-SFTs with no computable configurations (Original result by Hanf-Myers 1974)
- Classifying the entropies of $\mathbb{Z}^2$-SFTs (Original result by Hochman-Meyerovitch 2010)

Two new results in general groups

Let $T : G \curvearrowright X \subset \{0,1\}^A$ be an effectively closed action of a finitely generated group.

Theorem (B-Sablik, 2016)

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Theorem (B, 2017)

For any pair of infinite and finitely generated groups H_1, H_2 there exists a $(G \times H_1 \times H_2)$-SFT such that its G-subaction is an extension of T.

How does one prove such a thing?

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$\Psi : \{0,1\}^{\mathbb{N}} \to \{0,1,\$\}^{\mathbb{Z}}$ given by:

$$\Psi(x)_j = \begin{cases} x_n & \text{if } j = 3^n \mod 3^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

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Example

If we write $x = x_0x_1x_2x_3\dots$ we obtain,

$$\Psi(x) = \dots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\dots$$

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How does one prove such a thing?

▷ pick afinite set of generators S of G.
▷ construct a subshift Π where every configuration is an S-tuple of configurations of the previous form.

$$S = \{1_G, s_1, \dots s_n\}$$

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Claim

If T is an effectively closed action, Π is effectively closed.

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▷ Take Π and construct a sofic $\mathbb{Z}^2$ subshift $\widetilde{\Pi}$ having Π in every horizontal row using the expansive simulation theorem.
▷ Using the decoding argument, construct a map from Π to X.
▷ Put in every G-coset of $G \times \mathbb{Z}^2$ a configuration of $\widetilde{\Pi}$.

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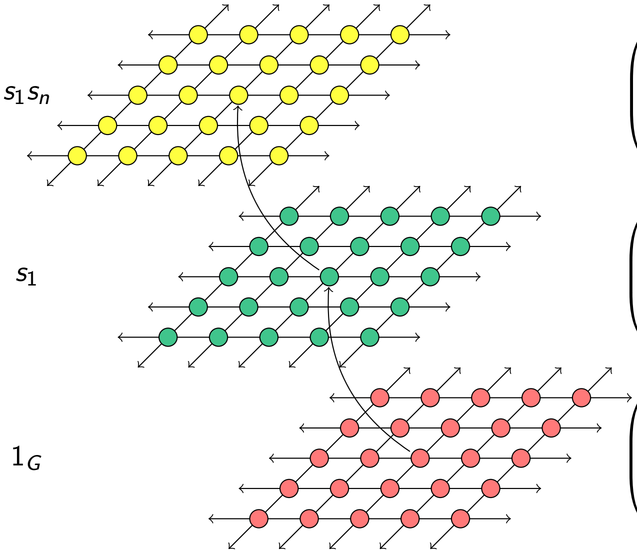

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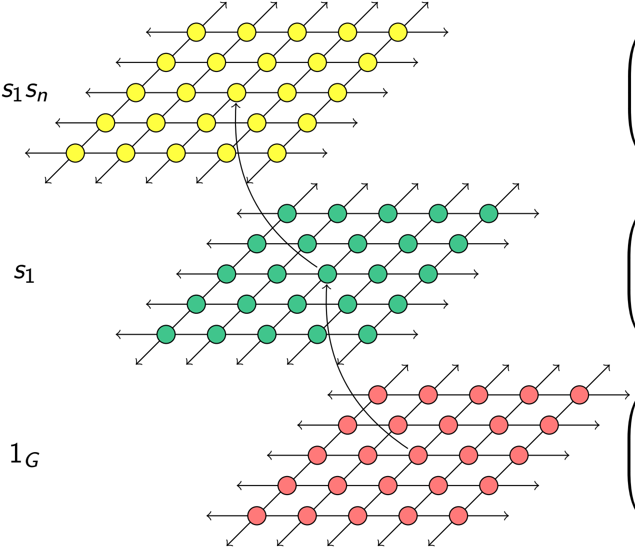

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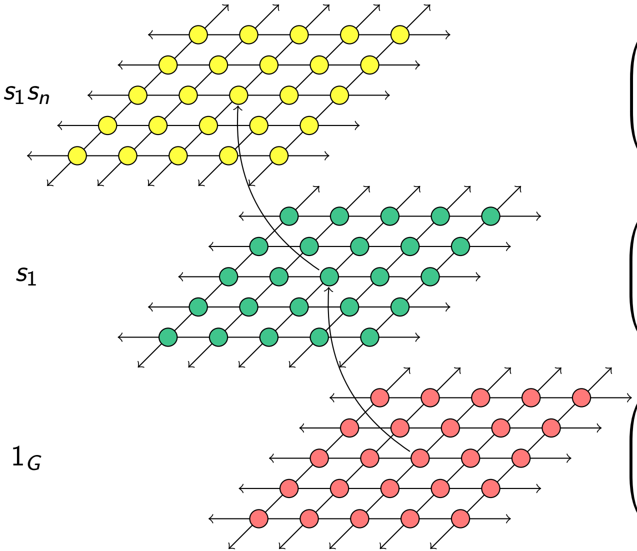

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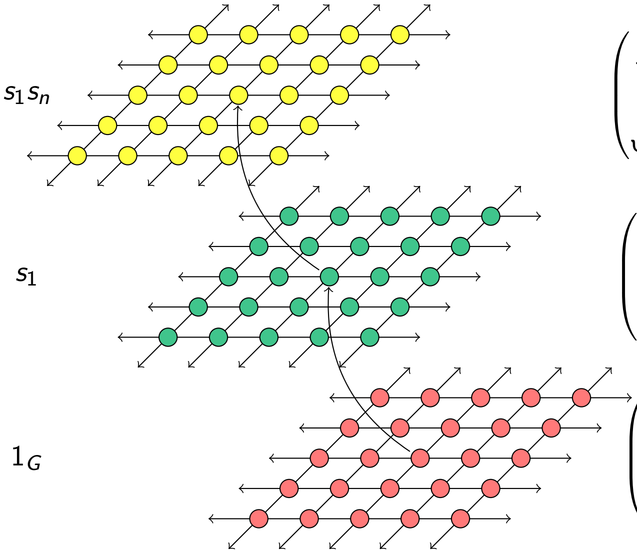

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Two corollaries

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If G is finitely generated, $\mathtt{WP}(G)$ is decidable and $d > 1$. Then $G \rtimes \mathbb{Z}^d$ admits a SA SFT.

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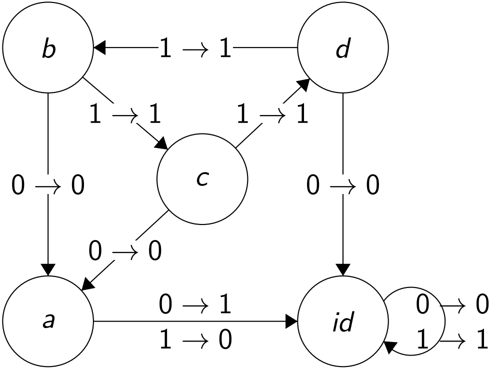
Theorem (B 2017)

If G_i are at least three infinite and finitely generated groups with decidable word problem. Then $G_1 \times \cdots \times G_n$ admits a SA SFT.

What about the Grigorchuk group?



The Grigorchuk group is generated by the actions a, b, c, d over $\{0,1\}^{\mathbb{N}}$.

What about the Grigorchuk group?

- The Grigorchuk group is infinite and finitely generated.
- It contains no copy of $\mathbb{Z}$ as a subgroup. For every $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = 1_G$.
- Decidable word problem (and conjugacy problem).
- It has intermediate growth.
- It is commensurable to its square. ie: G and $G \times G$ have an isomorphic finite index subgroup.

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The Grigorchuk group admits a strongly aperiodic SFT.

Thank you for your attention!